

Division Theorems for Exact Sequences

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Abstract. Under certain integrability and geometric conditions, we prove division theorems for the exact sequences of holomorphic vector bundles and improve the results in the case of Koszul complex. By introducing a singular Hermitian structure on the trivial bundle, our results recover Skoda's division theorem for holomorphic functions on pseudoconvex domains in complex Euclidean spaces.

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Introduction

A classical problem in complex and algebraic geometry is to characterize the image of global holomorphic sections under a sheaf-homomorphism. Let E, E' be holomorphic vector bundles over a complex manifold, and $\Phi : E \rightarrow E'$ be a holomorphic homomorphism, it is interesting to characterize the image of the induced homomorphism on cohomology groups

$$H^q(M, \Omega^p(E)) \rightarrow H^q(M, \Omega^p(E'))$$

under effective integrability and differential-geometric conditions. Skoda's Division theorem ([S72,S78,D82]) is one of the fundamental result on this kind of questions. Skoda's theorem has played important roles in many important work in algebraic geometry. Siu used Skoda's theorem to establish the deformation invariance of plurigenera and prove the finite generation of canonical ring of compact complex algebraic manifolds of general type([Siu 98,00,04,05,07]). Skoda's theorem is an analogue of Hilbert's Nullstellensatz, but the remarkable feature of effectiveness makes it very powerful. In [B87, Laz99], the authors used Skoda's theorem to prove effective versions of the Nullstellensatz.

The statement of Skoda's theorem is the following: Let Ω be a pseudoconvex domain in \mathbb{C}^n , $\psi \in \text{PSH}(\Omega)$ (the set of plurisubharmonic functions on Ω), $g_1 \cdots, g_r \in \mathcal{O}(\Omega)$ (the set of holomorphic functions on Ω),

then for every $f \in \mathcal{O}(\Omega)$ with $\int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV < +\infty$, there exist holomorphic functions $h_1, \dots, h_r \in \mathcal{O}(\Omega)$ such that $f = \sum g_i h_i$ holds on Ω and $\int_{\Omega} |h|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi} dV \leq \frac{1+\varepsilon}{\varepsilon} \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV$ where $|g|^2 = \sum_i |g_i|^2$, $|h|^2 = \sum_i |h_i|^2$, $q = \min\{n, r-1\}$ and $\varepsilon > 0$ is a constant. In [S78, D82], this theorem was generalized to (generic) surjective homomorphisms between holomorphic vector bundles by solving $\bar{\partial}$ -equations. The surjectivity was used in two places in [S78] and [D82]. One is to construct a smooth splitting map which reduces the question under consideration to $\bar{\partial}$ -equations, another is to relate an arbitrary Hermitian structure on the quotient bundle to the quotient Hermitian structure induced by the surjective homomorphism (actually this was also done by constructing a smooth splitting in a more general case). A recent generalization of Skoda's theorem was given in [V08] for line bundles where the notion of Skoda triple was introduced, by choosing Skoda triples the author established various division theorems.

In this paper, we consider the question of establishing division theorems in more general settings. In section 4, we will prove division theorems for a generically exact sequence of holomorphic vector bundles (theorem 4.2, corollaries 4.3, 4.4) and division theorems for a single homomorphism (theorem 4.2'). Moreover, in section 5, for the case of the Koszul complex which involves nonsurjective homomorphisms, we improve the result for general exact sequences (theorem 5.4 and corollary 5.5).

As a corollary, we get a division theorem for Koszul complex over a pseudoconvex domain in \mathbb{C}^n . More precisely, let Ω be a domain in \mathbb{C}^n , $g_1, \dots, g_r \in \mathcal{O}(\Omega)$, we define a sheaf-homomorphism for each $1 \leq \ell \leq r$ by setting

$$\begin{aligned} \wedge^{\ell} \mathcal{O}_{\Omega}^{\oplus r} &\rightarrow \wedge^{\ell-1} \mathcal{O}_{\Omega}^{\oplus r} \\ (h_{i_1 \dots i_{\ell}})_{i_1 \dots i_{\ell}=1}^r &\mapsto (f_{i_1 \dots i_{\ell-1}})_{i_1 \dots i_{\ell-1}=1}^r \quad \text{with } f_{i_1 \dots i_{\ell-1}} = \sum_{1 \leq \nu \leq r} g_{\nu} h_{\nu i_1 \dots i_{\ell-1}}. \end{aligned}$$

By choosing appropriate Hermitian structure and the standard argument of weak compactness, we show that Skoda's theorem is exactly the special case where $\ell = 1$ of the following result (see corollary 5.7) about the homomorphisms defined in this way.

Let Ω be a pseudoconvex domain in \mathbb{C}^n , $g_1, \dots, g_r \in \mathcal{O}(\Omega)$, $\psi \in \text{PSH}(\Omega)$ and $\varepsilon > 0$ a constant, then for every global section $(f_{i_1 \dots i_{\ell-1}})_{i_1 \dots i_{\ell-1}=1}^r \in \Gamma(\Omega, \wedge^{\ell-1} \mathcal{O}_{\Omega}^{\oplus r})$ ($1 \leq \ell \leq r$) satisfying

$$\sum_{1 \leq \nu \leq r} g_{\nu} f_{\nu i_1 \dots i_{\ell-2}} = 0 \quad \text{and} \quad \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV < +\infty,$$

there exists at least one $(h_{i_1 \dots i_\ell})_{i_1 \dots i_\ell=1}^r \in \Gamma(\Omega, \wedge^\ell \mathcal{O}_\Omega^{\oplus r})$ such that

$$f_{i_1 \dots i_{\ell-1}} = \sum_{1 \leq \nu \leq r} g_\nu h_{\nu i_1 \dots i_{\ell-1}},$$

and

$$\int_\Omega |h|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi} dV \leq \frac{1+\varepsilon}{\varepsilon} \int_\Omega |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV,$$

where $|g|^2 = \sum_i |g_i|^2$, $|h|^2 = \sum_{i_1 < \dots < i_\ell} |h_{i_1 \dots i_\ell}|^2$, $|f|^2 = \sum_{i_1 < \dots < i_{\ell-1}} |f_{i_1 \dots i_{\ell-1}}|^2$, $q = \min\{n, r-1\}$.

In the surjective case, we also give sufficient conditions for the solvability of division problem in matrix form. This is achieved by applying Skoda's original division theorem and a purely algebraic argument (see corollary 5.9). Our integrability conditions are comparable with those conditions appeared in [KT71].

As an example of the application of Spencer's trick, we could construct smooth lifting for an exact sequence and reduce the question to solving $\bar{\partial}$ -equations with additional restrictions, the main difficulty would be that it requires to look for solutions of $\bar{\partial}$ -equations which take values in a subsheaf. Although we don't use this fact essentially in our proof of division theorems, we will sketch the rough ideal in section 4.

Instead of solving $\bar{\partial}$ -equations, we hope to implement Skoda's original estimate in general case. To this end, we first formulate an algebraic inequality (lemma 2.2) which helps to complete square in the proof of our main estimate (lemma 3.2). To apply this algebraic lemma, we need to consider certain bundle homomorphisms, and the second fundamental form is quite useful which is different from that used in [S78] and [D82]. Since the bundle homomorphism is surjective in [S78] and [D82], the second fundamental form of the kernel is well defined, but in more general case the kernel is only a subsheaf which is no longer a subbundle. For this reason, we consider the second fundamental form of the line bundle spanned by the homomorphism itself inside the holomorphic bundle $\text{Hom}(E, E')$ which is always well defined outside the subset of zeros of the bundle-homomorphism under consideration. To obtain our main estimate, we also use the twisted Bochner-Kodaira-Nakano formula which is quite natural and useful in geometry and analysis. Applications of this kind of technique could be found in many references, e.g. [OT], [Siu82], [Siu00], [Siu04] and [Dr08]. The discussion about the division problem of Koszul complex via the residue theory and its interesting applications could be found in [A04], [A06] and [AG10].

1. The Second Fundamental Form

Let (M, ω) be a Kähler manifold, $(E, h), (E', h')$ Hermitian holomorphic vector bundles over M , $\dim_{\mathbb{C}} M = n, \text{rank}_{\mathbb{C}} E = r, \text{rank}_{\mathbb{C}} E' = r'$. $\Phi : \mathcal{O}_M(E) \rightarrow \mathcal{O}_M(E')$ is a sheaf-homomorphism, given by a holomorphic section $\Phi \in \Gamma(M, \text{Hom}(E, E'))$.

The homomorphism Φ is assumed, for simplicity, to be nowhere zero on M in sections 1 and 2.

We shall adopt the following convention on the range of indices.

$$1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq r, 1 \leq a, b \leq r'.$$

We also adhere to the summation convention that sum is performed over strictly increasing multi-indices.

By using local coordinates z_1, \dots, z_n of M , holomorphic frames $\{e_1, \dots, e_r\}, \{e'_1, \dots, e'_{r'}\}$ of E and E' respectively we write

$$\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz_{\alpha} \wedge d\bar{z}_{\beta}, h = h_{i\bar{j}} e_i^* \otimes \bar{e}_j^*, h' = h'_{a\bar{b}} e_a'^* \otimes \bar{e}_b'^*,$$

and

$$\Phi = \Phi_{ia} e_a' \otimes e_i^*. \quad (1)$$

where $\{e_1^*, \dots, e_r^*\}$ and $\{e_1'^*, \dots, e_{r'}'^*\}$ are dual frames of E^* and E'^* respectively. We also know by definition that the adjoint homomorphism of Φ w.r.t. the Hermitian structures h and h' is given by

$$\Phi^* = \overline{\Phi_{ia}} h^{\bar{i}j} h'_{a\bar{b}} e_j \otimes e_b'^* \quad (2)$$

where $(h^{\bar{i}j}) = (h_{i\bar{j}})^{-1}$.

It is convenient to make it a convention that we always work with normal coordinates and normal frames for a pointwise computation, i.e. we have $g_{\alpha\bar{\beta}} = \delta_{\alpha\beta}, h_{i\bar{j}} = \delta_{ij}, h'_{a\bar{b}} = \delta_{ab}$ and $dg_{\alpha\bar{\beta}} = dh_{i\bar{j}} = dh'_{a\bar{b}} = 0$ at the point under consideration.

We define a smooth section B of $A^{1,0}(\text{Hom}(E, E'))$ as the differentiation of Φ , i.e. for every $X \in T^{1,0}M$

$$BX = \nabla_X^{\text{Hom}(E, E')} \Phi \quad (3)$$

where $\nabla^{\text{Hom}(E, E')}$ is the induced Chern connection on $\text{Hom}(E, E')$.

Let B^* be the adjoint of B where B is viewed as a homomorphism from $T^{1,0}M$ to $\text{Hom}(E, E')$. By definition, we know B^* is a smooth section of $\text{Hom}(\text{Hom}(E, E'), T^{1,0}M)$. A pointwise computation shows that the exterior differentiation of Φ^* and B^* are related as follows

$$\bar{\partial}\Phi^* = (B^*e_i^* \otimes e_a')^{\flat} \otimes e_i \otimes e_a'^* \quad (4)$$

where \flat is the \mathbb{C} -linear map of lowering indices by using the metric $g_{\alpha\bar{\beta}}$.

Now we compute the derivatives up to second order of the function $\varphi = \log \|\Phi\|^2$ in terms of the homomorphism B . Since Φ is holomorphic, we have

$$\begin{aligned} \partial_{\alpha}\varphi &= \|\Phi\|^{-2} (\nabla_{\alpha}^{\text{Hom}(E, E')} \Phi, \Phi) \\ &= e^{-\varphi} (B \frac{\partial}{\partial z_{\alpha}}, \Phi) \end{aligned} \quad (5)$$

$$\text{grad}^{0,1}\varphi = \|\Phi\|^{-2} g^{\bar{\alpha}\beta} (B \frac{\partial}{\partial z_{\beta}}, \Phi) \frac{\partial}{\partial \bar{z}_{\alpha}} \quad (6)$$

$$\begin{aligned} \partial_{\alpha}\bar{\partial}_{\bar{\beta}}\varphi &= e^{-\varphi} [-\bar{\partial}_{\bar{\beta}}\varphi (B \frac{\partial}{\partial z_{\alpha}}, \Phi) + (\nabla_{\bar{\beta}}^{\text{Hom}(E, E')} \nabla_{\alpha}^{\text{Hom}(E, E')} \Phi, \Phi) \\ &\quad + (B \frac{\partial}{\partial z_{\alpha}}, \nabla_{\bar{\beta}}^{\text{Hom}(E, E')} \Phi)] \\ &= e^{-\varphi} [-\|\Phi\|^{-2} \overline{(B \frac{\partial}{\partial z_{\beta}}, \Phi)} (B \frac{\partial}{\partial z_{\alpha}}, \Phi) + (B \frac{\partial}{\partial z_{\alpha}}, B \frac{\partial}{\partial z_{\beta}}) \\ &\quad + (F_{\bar{\beta}\alpha}^{\text{Hom}(E, E')} \Phi, \Phi)] \end{aligned} \quad (7)$$

where $F_{\bar{\beta}\alpha}^{\text{Hom}(E, E')} = \left[\nabla_{\bar{\beta}}^{\text{Hom}(E, E')}, \nabla_{\alpha}^{\text{Hom}(E, E')} \right]$ is the curvature of the induced Chern connection on $\text{Hom}(E, E')$.

Let P be the orthonormal projection from $\text{Hom}(E, E')$ onto the subbundle $\text{Span}_{\mathbb{C}}\{\Phi\}^{\perp} \subseteq \text{Hom}(E, E')$, we define

$$B_{\Phi} = P \circ B : T^{1,0}M \rightarrow \text{Span}_{\mathbb{C}}\{\Phi\}^{\perp}. \quad (8)$$

By definition B_{Φ} is the second fundamental form of the holomorphic line bundle $\text{Span}_{\mathbb{C}}\{\Phi\}$ in $\text{Hom}(E, E')$.

Let L be a Hermitian holomorphic line bundle over M , $\{\sigma\}$ be a holomorphic local frame of L , we denote its dual frame by $\{\sigma^*\}$. For any $v = v_{\bar{K}i} dz \wedge d\bar{z}_K \otimes \sigma \otimes e_i \in A^{n,k}(L \otimes E)$, we define the associated smooth section A_v of $\text{Hom}(\wedge^{n,k-1}TM \otimes L^* \otimes E^*, T^{1,0}M)$ by setting

$$A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*) = (-1)^n g^{\bar{\alpha}\beta} v_{\bar{\alpha}Ki} \frac{\partial}{\partial z_{\beta}}, \quad (9)$$

for $1 \leq k \leq n$, $|K| = k - 1$, here we denote

$$dz = dz_1 \wedge \cdots \wedge dz_n \text{ and } \frac{\partial}{\partial z} = \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}.$$

It is easy to see A_v is well defined. Moreover, if $\{z_1, \dots, z_n\}$, $\{e_1, \dots, e_r\}$, $\{e'_1, \dots, e'_{r'}\}$ and $\{\sigma\}$ are normal at the point $x \in M$, then we have

$$(A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*), X) = (v, X^b \wedge dz \wedge d\bar{z}_K \otimes \sigma \otimes e_i) \quad (10)$$

for every $X \in T_x^{1,0}M$ and multi-index K with $|K| = k - 1$.

Multiplying both sides of the equality (7) by $v_{\alpha K i} \overline{v_{\beta K i}}$ and summing over α, β, i and increasing multi-indices K with $|K| = k - 1$ give the following expression of $\partial_\alpha \partial_{\bar{\beta}} \varphi v_{\alpha K i} \overline{v_{\beta K i}}$ which will be used to handle the curvature term in the Bochner-Kodaira-Nakano formula.

$$\begin{aligned} \partial_\alpha \partial_{\bar{\beta}} \varphi v_{\alpha K i} \overline{v_{\beta K i}} &= e^{-\varphi} [-|(BA_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*), \frac{\Phi}{\|\Phi\|})|^2 \\ &\quad + \left\| BA_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*) \right\|^2 \\ &\quad + (F_{A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*) A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)}^{\text{Hom}(E, E')} \Phi, \Phi)] \\ &= e^{-\varphi} \|B_\Phi A_v\|^2 \\ &\quad - e^{-\varphi} (F_{A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*) A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)}^{\text{Hom}(E, E')} \Phi, \Phi). \quad (11) \end{aligned}$$

2. Algebraic Preliminaries

In this section we introduce the notion of trace which generalizes the usual conception of trace for linear transformations. By using the Cauchy-Schwarz inequality, we will establish a fundamental estimate for the generalized trace which plays an important role in our main estimate. In order to apply this inequality, we also collect in this section the pointwise calculations concerning the smooth section $\text{Tr} B_\Phi A \in \Gamma(\wedge_M^{n, k-1} \otimes L \otimes E')$. Lemma 2.2 and lemma 2.3 make up the major algebraic part of the proof of our main estimate. Its geometric ingredient is the twisted Bochner-Kodaira-Nakano formula (combined with the Morrey's trick) on a bounded domain which will be discussed in section 3.

Definition 2.1. Let U, V, W be linear spaces, $D \in \text{Hom}(U, V)$, $\rho : V \times U^* \rightarrow W$ a bilinear map. If U is finite-dimensional we define the trace $\text{Tr}_\rho D \in W$ of the linear map D w.r.t. ρ to be

$$\mathrm{Tr}_\rho D = \sum_i \rho(Du_i, u^i) \quad (12)$$

where $\{u_i\}$ is a basis of U , $\{u^i\} \subseteq U^*$ is its dual basis.

The definition of $\mathrm{Tr}_\rho D$ is obviously independent of the choice of the basis $\{u_i\}$. Now we establish the basic estimate for $\mathrm{Tr}_\rho D$.

Lemma 2.2. If U, V, W are Hermitian spaces, $D \in \mathrm{Hom}(U, V)$ and $\rho : V \times U^* \rightarrow W$ is a bilinear map, then we have

$$\|\mathrm{Tr}_\rho D\|_W \leq \sqrt{\mathrm{rank}(D)} \|\rho\| \|D\|. \quad (13)$$

Proof. By using the singular value decomposition theorem for a linear map between Hermitian spaces, one can always find an orthonormal basis $\{u_i\}$ of U such that

$$(Du_i, Du_j)_V = 0 \text{ for } i \neq j.$$

Since $\mathrm{Im}(D) = \mathrm{Span}_{\mathbb{C}}\{Du_i\}$ and Du_1, Du_2, \dots are mutually perpendicular, we have

$$\sharp\{i \mid Du_i \neq 0\} = \dim(\mathrm{Im}(D)) = \mathrm{rank}(D).$$

Let $\{u^i\} \subseteq U^*$ be the dual basis of $\{u_i\}$, then $\{u^i\}$ forms a orthonormal basis of U^* , i.e. $(u^i, u^j) = \delta_{ij}$. Consequently, by using the Cauchy-Schwarz inequality and the definition of $\|\rho\|$:

$$\|\rho\| := \max_{\substack{\|v\|_V=1 \\ \|\alpha\|_{U^*}=1}} \|\rho(v, \alpha)\|_W \text{ and } \|D\| = \sqrt{\sum_i \|Du_i\|_V^2},$$

we get the desired estimate as follows

$$\begin{aligned} \|\mathrm{Tr}_\rho D\|_W^2 &= \left\| \sum_i \rho(Du_i, u^i) \right\|_W^2 \\ &\leq \left(\sum_i \|\rho(Du_i, u^i)\|_W \right)^2 \\ &\leq \|\rho\|^2 \left(\sum_i \|Du_i\|_V \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \|\rho\|^2 \left(\sum_{Du_i \neq 0} \|Du_i\|_V \right)^2 \\
&\leq \text{rank}(D) \|\rho\|^2 \sum_{Du_i \neq 0} \|Du_i\|_V^2 \\
&= \text{rank}(D) \|\rho\|^2 \|D\|^2,
\end{aligned}$$

which completes the proof. \square

We will apply lemma 2.2 in two specific circumstances. In the following lemma 2.3, we will choose $V = W \otimes U$ and $\rho : W \otimes U \times U^* \rightarrow W$ to be the natural contraction between U and its dual space U^* . If we identify U and U^* via the \mathbb{C} -antilinear isomorphism defined by the Hermitian structure on U , then we have explicitly

$$\text{Tr}_\rho D = (Du_i, w_a \otimes u_i)_{W \otimes U} w_a$$

for orthonormal bases $\{u_i\} \subseteq U, \{w_a\} \subseteq W$.

In section 5, we will consider $U = \wedge^p V$, and $\rho : V \times \wedge^p V^* \rightarrow \wedge^{p-1} V^*$ defined by the interior product, i.e.

$$\rho(v, \xi) := v \lrcorner \xi.$$

Obviously, the Cauchy-Schwarz inequality shows that $\|\rho(v, u^*)\|_W \leq \|v\|_V \|u^*\|_{U^*}$, we get therefore $\|\rho\| \leq 1$ in both of the cases mentioned above. Since we always work with specific bilinear map, the subscript ρ will be omitted without causing ambiguity. The importance of the inequality (13) is that the coefficient of the left hand side only depends on the rank of D .

Now we proceed to prove the key identity involving $\text{Tr} B_\Phi A$. Since the computations in this section are pointwise, as mentioned before, we will work with normal coordinates and normal frames at a given point.

For a given $v \in A^{n,k}(L \otimes E), 1 \leq k \leq n$, we define by (9) the associated homomorphism $A_v \in \text{Hom}(\wedge^{n,k-1} TM \otimes L^* \otimes E^*, T^{1,0} M)$. Under the standard bundle isomorphism

$$\begin{aligned}
&\text{Hom}(\wedge^{n,k-1} TM \otimes L^* \otimes E^*, \text{Hom}(E, E')) \\
&\cong \text{Hom}(E^*, \wedge^{n,k-1} M \otimes L \otimes E' \otimes E^*),
\end{aligned} \tag{14}$$

we could define $\text{Tr} B_\Phi A_v \in \Gamma(\wedge^{n,k-1} M \otimes L \otimes E')$ by (12) where the bilinear map ρ is given by the pairing between E and E^* which is defined by the Hermitian structure on E .

The main result of about $\text{Tr} B_\Phi A_v$ is recorded in the following formula.

Lemma 2.3. For any u in $\wedge^{n,k-1} M \otimes L \otimes E'$ and v in $\wedge^{n,k} M \otimes L \otimes E$, we have

$$(\bar{\partial}\Phi^* \wedge u, v) - (\Phi^* u, \text{grad}^{0,1} \varphi \lrcorner v) = (u, \text{Tr} B_\Phi A_v) \quad (15)$$

where $\varphi = \log \|\Phi\|$, $\Phi \in \Gamma(M, \text{Hom}(E, E'))$ and A_v is defined by v as described in (9), $1 \leq k \leq n$.

Proof. Let $u = u_{\overline{K}a} dz \otimes d\bar{z}_K \otimes \sigma \otimes e'_a \in \wedge^{n,k-1} M \otimes L \otimes E'$, $v = v_{\overline{J}i} dz \otimes d\bar{z}_J \otimes \sigma \otimes e_i \in \wedge^{n,k} M \otimes L \otimes E$. We know by the definition (12) and the identification (14) that

$$\begin{aligned} \text{Tr} B_\Phi A_v &= (e_i^* \lrcorner B_\Phi A_v, dz \otimes d\bar{z}_K \otimes \sigma \otimes e'_a \otimes e_i^*) dz \otimes d\bar{z}_K \otimes \sigma \otimes e'_a \\ &= (B_\Phi A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*), e'_a \otimes e_i^*) dz \otimes d\bar{z}_K \otimes \sigma \otimes e'_a, \end{aligned}$$

which implies that

$$\begin{aligned} (u, \text{Tr} B_\Phi A_v) &= u_{\overline{K}a} (dz \otimes d\bar{z}_K \otimes \sigma \otimes e'_a, \text{Tr} B_\Phi A_v) \\ &= u_{\overline{K}a} (e'_a \otimes e_i^*, B_\Phi A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)). \end{aligned} \quad (16)$$

From the equalities (4) and (10), it follows that

$$\begin{aligned} (\bar{\partial}\Phi^* \wedge u, v) &= ((B^*(e'_a \otimes e_i^*))^b \otimes e_i \otimes e_a'^*(u), v) \\ &= u_{\overline{K}b} ((B^*(e'_a \otimes e_i^*))^b \wedge dz \otimes d\bar{z}_K \otimes \sigma \otimes e_a'^*(e_b') e_i, v) \\ &= u_{\overline{K}a} ((B^*(e'_a \otimes e_i^*))^b \wedge dz \otimes d\bar{z}_K \otimes \sigma \otimes e_i, v) \\ &= u_{\overline{K}a} (B^*(e'_a \otimes e_i^*), A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)) \\ &= u_{\overline{K}a} (e'_a \otimes e_i^*, B A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)). \end{aligned} \quad (17)$$

We also obtain from (6) that

$$\begin{aligned} &(\Phi^* u, \text{grad}^{0,1} \varphi \lrcorner v) \\ &= (\Phi^* u, \|\Phi\|^{-2} (B_{\frac{\partial}{\partial z_\alpha}}, \Phi) \frac{\partial}{\partial \bar{z}_\alpha} \lrcorner v) \\ &= (-1)^n (\Phi^* u, \|\Phi\|^{-2} (B_{\frac{\partial}{\partial z_\alpha}}, \Phi) v_{\overline{\alpha K}i} dz \otimes d\bar{z}_K \otimes \sigma \otimes e_i) \\ &= (\Phi^* u, \|\Phi\|^{-2} (B A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*), \Phi) dz \otimes d\bar{z}_K \otimes \sigma \otimes e_i) \\ &= (u, (B A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*), \frac{\Phi}{|\Phi|}) dz \otimes d\bar{z}_K \otimes \sigma \otimes \frac{\Phi}{|\Phi|}(e_i)) \\ &= (u, \frac{\Phi_{ib}}{|\Phi|} (B A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*), \frac{\Phi}{|\Phi|}) dz \otimes d\bar{z}_K \otimes \sigma \otimes e_b') \\ &= u_{\overline{K}a} \frac{\Phi_{ia}}{|\Phi|} (\frac{\Phi}{|\Phi|}, B A_v (\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)) \end{aligned}$$

$$= u_{\overline{K}a}(e'_a \otimes e_i^*, QBA_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}_K} \otimes \sigma^* \otimes e_i^*)) \quad (18)$$

where Q is the orthonormal projection from $E' \otimes E^*$ onto the line bundle $\text{Span}_{\mathbb{C}}\{\Phi\}$. Now (15) follows from (16) (17) (18) and the definition (8) of B_{Φ} , the proof is complete. \square

3. The Main Estimate

Notations. We introduce some notations which are needed to simplify our statements. Given a measurable function μ on a Kähler manifold (M, ω) , we define the associated signed measure by setting

$$dV_{\mu} = \mu dV_{\omega} \quad (19)$$

where $dV_{\omega} = \frac{\omega^n}{n!}$ is the volume form of ω . Let Ω be a domain in M . We denote by

$$(\cdot, \cdot)_{\Omega, \mu}, \quad \|\cdot\|_{\Omega, \mu}$$

the L^2 -inner product and L^2 -norm defined by using the measure dV_{μ} . When μ is nonnegative, the corresponding Hilbert space of square integrable (n, k) -forms on Ω valued in $L \otimes E$ and $L \otimes E'$ will be denoted respectively by $L^2_{n,k}(\Omega, L \otimes E, dV_{\mu})$ and $L^2_{n,k}(\Omega, L \otimes E', dV_{\mu})$ respectively. The subscript μ in $(\cdot, \cdot)_{\Omega, \mu}$ will be dropped for $\mu = 1$.

We recall the definition of m -tensor positivity.

Definition 3.1. A Hermitian holomorphic vector bundle (E, h) is said to be m -tensor semi-positive(semi-negative) if the curvature F (of Chern connection) satisfies $\sqrt{-1}F(\eta, \eta) \geq 0$ (≤ 0) for every $\eta = \eta_{\alpha i} \frac{\partial}{\partial z_{\alpha}} \otimes e_i \in T^{1,0}M \otimes E$ with $\text{rank}(\eta_{\alpha i}) \leq m$ where z_1, \dots, z_n are holomorphic coordinates of M , $\{e_1, \dots, e_r\}$ is a holomorphic frame of E and m is a positive integer. In this case, we write $E \geq_m 0$ ($E \leq_m 0$).

It is easy to see the above definition does not depend on the choice of the holomorphic coordinates z_1, \dots, z_n or the holomorphic frame $\{e_1, \dots, e_r\}$.

Definition3.2. Let E be a holomorphic vector bundle over M , $Z \subsetneq M$ be a subvariety, and h be a Hermitian structure on $E|_{M \setminus Z}$. If for each $z \in Z$, there exist a neighborhood U of z , a smooth frame $\{e_1, \dots, e_r\}$ over U and some constant $\kappa > 0$ such that the matrix $[h_{i\bar{j}}(w) - \kappa\delta_{ij}]$ is semi-positive for every $w \in U \setminus Z$ where $h_{i\bar{j}} := h(e_i, e_j)$ and δ_{ij} is the Kronecker delta, then we call h a singular Hermitian structure on E which has singularities in Z .

Let $\Omega \Subset M$ be a domain with smooth boundary, $\rho \in C^\infty(\overline{\Omega})$ a defining function of Ω , i.e. Ω is given by $\rho < 0$ and $d\rho \neq 0$ on $\partial\Omega$. Ω is said to be pseudoconvex if the levi form L_ρ is semi-positive on $T^{1,0}\partial\Omega$. This condition is independent of the choice of the defining function.

Now we are in the position to prove the main estimate:

Lemma3.3. Let (M, ω) be a Kähler manifold, and let E be a Hermitian holomorphic vector bundle over M , L a Hermitian holomorphic line bundle over M . The Hermitian structures of these bundles may have singularity along $\Phi^{-1}(0)$ and $\Omega \Subset M \setminus \Phi^{-1}(0)$ is a pseudoconvex domain with smooth boundary. Assume that the following conditions hold on Ω :

1. $E \geq_m 0, m \geq \min\{n - k + 1, r\}, 1 \leq k \leq n$;
2. the curvature of $\text{Hom}(E, E')$ satisfies

$$(F_{X\overline{X}}^{\text{Hom}(E, E')} \Phi, \Phi) \leq 0 \text{ for every } X \in T^{1,0}M;$$

3. the curvature of L satisfies

$$\sqrt{-1}(\varsigma c(L) - \partial\bar{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \bar{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma + \delta)\partial\bar{\partial}\varphi.$$

Then the following estimate

$$\left\| |\Phi|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_{\Omega, \varsigma + \tau}^2 + \|\bar{\partial} v\|_{\Omega, \varsigma}^2 \geq \|u\|_{\Omega, \frac{\varsigma(\lambda\delta + \lambda\varsigma - \varsigma)}{(\varsigma + \delta)|\Phi|^2}}^2 \quad (20)$$

holds for every $\bar{\partial}$ -closed $u \in A^{n, k-1}(\overline{\Omega}, L \otimes E)$ satisfying $|\Phi^* u|^2 \geq \lambda|\Phi|^2|u|^2$ a.e.(w.r.t. dV_ω) on Ω and every $v \in A^{n, k}(\overline{\Omega}, L \otimes E) \cap \text{Dom}(\bar{\partial}^*)$, where $c(L)$ denotes the Chern form, $q = \max_{\Omega} \text{rank} B_\Phi, \varphi = \log |\Phi|^2, 0 < \varsigma \in C^\infty(\overline{\Omega})$ and λ, δ, τ are measurable functions on Ω satisfying $\lambda, \tau > 0, \varsigma + \delta \geq 0$. All the weighted norms are described at the beginning of this section.

Proof. Since we work on a fixed domain, the subscript Ω will be omitted in the following proof. We assume Ω is given by $\rho < 0$ and $d\rho \neq 0$ on $\partial\Omega$ where $\rho \in C^\infty(\overline{\Omega})$.

Step 1. From the condition

$$|\Phi^*u|^2 \geq \lambda|\Phi|^2|u|^2,$$

we get

$$\begin{aligned} & \left\| |\Phi|^{-2}\Phi^*u + \bar{\partial}^*v \right\|_{\Omega, \varsigma+\tau}^2 + \|\bar{\partial}v\|_{\Omega, \varsigma}^2 \\ &= \left\| \sqrt{\varsigma+\tau}|\Phi|^{-2}\Phi^*u \right\|^2 + 2\operatorname{Re}((\varsigma+\tau)|\Phi|^{-2}\Phi^*u, \bar{\partial}^*v) \\ & \quad + \left\| \sqrt{\varsigma+\tau}\bar{\partial}^*v \right\|^2 + \left\| \sqrt{\varsigma}\bar{\partial}v \right\|^2 \\ &= \left\| \sqrt{\varsigma}|\Phi|^{-2}\Phi^*u \right\|^2 + \left\| |\Phi|^{-2}\Phi^*u + \bar{\partial}^*v \right\|_\tau^2 \\ & \quad + 2\operatorname{Re}(\varsigma e^{-\varphi}\Phi^*u, \bar{\partial}^*v) + \left\| \sqrt{\varsigma}\bar{\partial}^*v \right\|^2 + \left\| \sqrt{\varsigma}\bar{\partial}v \right\|^2 \\ &\geq \|u\|_{|\Phi|^{-2}\lambda\varsigma}^2 + \left\| |\Phi|^{-2}\Phi^*u + \bar{\partial}^*v \right\|_\tau^2 \\ & \quad + 2\operatorname{Re}(\varsigma e^{-\varphi}\Phi^*u, \bar{\partial}^*v) + \left\| \sqrt{\varsigma}\bar{\partial}^*v \right\|^2 + \left\| \sqrt{\varsigma}\bar{\partial}v \right\|^2. \end{aligned} \quad (21)$$

From the condition that

$$v \in A^{n,k}(\overline{\Omega}, L \otimes E) \cap \operatorname{Dom}(\bar{\partial}^*),$$

we know

$$\operatorname{grad}^{0,1}\rho \lrcorner v = 0 \text{ on } \partial\Omega.$$

By using the twisted Bochner-Kodaira-Nakano formula and Morrey's trick, it follows from the pseudoconvexity of Ω that (see [Siu82], [Siu00])

$$\begin{aligned} & \left\| \sqrt{\varsigma}\bar{\partial}^*v \right\|^2 + \left\| \sqrt{\varsigma}\bar{\partial}v \right\|^2 \\ &= \int_\Omega \varsigma |\bar{\nabla}v|^2 + \varsigma([\sqrt{-1}F^{L \otimes E}, \Lambda_\omega]v, v) - \nabla^{\bar{\alpha}}\nabla^\beta \varsigma(\frac{\partial}{\partial \bar{z}_\alpha} \lrcorner v, \frac{\partial}{\partial \bar{z}_\beta} \lrcorner v) \\ & \quad + 2\operatorname{Re}(\operatorname{grad}^{0,1}\varsigma \lrcorner v, \bar{\partial}^*v)dV_\omega + \int_{\partial\Omega} \varsigma \nabla^{\bar{\alpha}}\nabla^\beta \rho(\frac{\partial}{\partial \bar{z}_\alpha} \lrcorner v, \frac{\partial}{\partial \bar{z}_\beta} \lrcorner v) \\ & \quad + (\bar{\partial}\rho \wedge \operatorname{grad}^{0,1}\varsigma \lrcorner v - \varsigma \bar{\partial}\rho \wedge \bar{\partial}^*v - \varsigma \bar{\partial}(\operatorname{grad}^{0,1}\rho \lrcorner v), v)dA \\ &= \int_\Omega \varsigma |\bar{\nabla}v|^2 + \varsigma([\sqrt{-1}F^{L \otimes E}, \Lambda_\omega]v, v) - \nabla^{\bar{\alpha}}\nabla^\beta \varsigma(\frac{\partial}{\partial \bar{z}_\alpha} \lrcorner v, \frac{\partial}{\partial \bar{z}_\beta} \lrcorner v) \\ & \quad + 2\operatorname{Re}(\operatorname{grad}^{0,1}\varsigma \lrcorner v, \bar{\partial}^*v)dV_\omega + \int_{\partial\Omega} \varsigma \nabla^{\bar{\alpha}}\nabla^\beta \rho(\frac{\partial}{\partial \bar{z}_\alpha} \lrcorner v, \frac{\partial}{\partial \bar{z}_\beta} \lrcorner v)dA \\ &\geq \int_\Omega \varsigma([\sqrt{-1}F^{L \otimes E}, \Lambda_\omega]v, v) - \nabla^{\bar{\alpha}}\nabla^\beta \varsigma(\frac{\partial}{\partial \bar{z}_\alpha} \lrcorner v, \frac{\partial}{\partial \bar{z}_\beta} \lrcorner v) \\ & \quad + 2\operatorname{Re}(\operatorname{grad}^{0,1}\varsigma \lrcorner v, \bar{\partial}^*v)dV_\omega, \end{aligned} \quad (22)$$

where $F^{L \otimes E}$ is the curvature of the Chern connection on $L \otimes E$, Λ_ω is the dual Lefschetz operator of the Kähler form ω , and dA is the induced volume form on $\partial\Omega$.

Step 2. In order to obtain pointwise the lower bound of the integrand of (22), we fix a point $x \in M$ and choose the local frames $\{e_1, \dots, e_r\}$, $\{\sigma\}$ of E and L respectively such that $(e_i, e_j) = \delta_{ij}$, $|\sigma|^2 = 1$ at x . The following pointwise computations are carried out at this fixed point x .

Set $F_{\alpha\bar{\beta}i\bar{j}}^{L \otimes E} = (F_{\alpha\bar{\beta}}^{L \otimes E} \sigma \otimes e_i, \sigma \otimes e_j)$, then we have

$$\begin{aligned} ([\sqrt{-1}F^{L \otimes E}, \Lambda_\omega] v, v) &= F_{\alpha\bar{\beta}i\bar{j}}^{L \otimes E} v_{\alpha K i} \overline{v_{\beta K j}} \\ &= F_{\alpha\bar{\beta}i\bar{j}}^E v_{\alpha K i} \overline{v_{\beta K j}} + F_{\alpha\bar{\beta}}^L v_{\alpha K i} \overline{v_{\beta K i}} \end{aligned}$$

where $v = v_{\bar{j}i} dz \wedge d\bar{z}_j \otimes \sigma \otimes e_i$, $F_{\alpha\bar{\beta}i\bar{j}}^E = (F_{\alpha\bar{\beta}}^E e_i, e_j)$, $F_{\alpha\bar{\beta}}^L = (F_{\alpha\bar{\beta}}^L \sigma, \sigma)$, and F^E, F^L are the curvature tensors of the Hermitian bundles E and L .

Since

$$E \geq_m 0, m \geq \min\{n - k + 1, r\} \text{ and } v_{\alpha K i} = 0 \text{ for } \alpha \in K,$$

we know by definition 3.1

$$F_{\alpha\bar{\beta}i\bar{j}}^E v_{\alpha K i} \overline{v_{\beta K j}} \geq 0.$$

From the condition

$$\sqrt{-1}(\zeta c(L) - \partial\bar{\partial}\zeta - \tau^{-1}\partial\zeta \wedge \bar{\partial}\zeta) \geq \sqrt{-1}q(\zeta + \delta)\partial\bar{\partial}\varphi,$$

it follows that

$$([\sqrt{-1}F^{L \otimes E}, \Lambda_\omega] v, v) \geq (q(\zeta + \delta)\partial_\alpha \partial_{\bar{\beta}} \varphi + \partial_\alpha \partial_{\bar{\beta}} \zeta + \tau^{-1}\partial_\alpha \zeta \overline{\partial_{\bar{\beta}} \zeta}) v_{\alpha K i} \overline{v_{\beta K i}}.$$

Substituting the above estimate into (22), we get

$$\begin{aligned} \left\| \sqrt{\zeta} \bar{\partial}^* v \right\|^2 + \left\| \sqrt{\zeta} \bar{\partial} v \right\|^2 &\geq \int_\Omega (q(\zeta + \delta)\partial_\alpha \partial_{\bar{\beta}} \varphi + \tau^{-1}\partial_\alpha \zeta \overline{\partial_{\bar{\beta}} \zeta}) v_{\alpha K i} \overline{v_{\beta K i}} \\ &\quad + 2\text{Re}(\text{grad}^{0,1} \zeta \lrcorner v, \bar{\partial}^* v) dV_\omega \\ &\stackrel{(11)}{=} \int_\Omega q(\zeta + \delta) |\Phi|^{-2} \|B_\Phi A_v\|^2 - q(\zeta + \delta) |\Phi|^{-2} \\ &\quad \cdot (F_{A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*) A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)}^{\text{Hom}(E, E')} \Phi, \Phi) \\ &\quad + \tau^{-1} \partial_\alpha \zeta \overline{\partial_{\bar{\beta}} \zeta} v_{\alpha K i} \overline{v_{\beta K i}} + 2\text{Re}(\text{grad}^{0,1} \zeta \lrcorner v, \bar{\partial}^* v) dV_\omega \\ &\geq \left\| \sqrt{q(\zeta + \delta)} B_\Phi A_v \right\|_{|\Phi|^{-2}}^2 + 2\text{Re}(\text{grad}^{0,1} \zeta \lrcorner v, \bar{\partial}^* v) \\ &\quad + \left\| \text{grad}^{0,1} \zeta \lrcorner v \right\|_{\tau^{-1}}^2. \end{aligned} \tag{23}$$

Step 3. To deal with the third term in (21), we first do integration by parts and then apply the Hölder inequality with an appropriate parameter. Integration by parts yields that

$$\begin{aligned}
(e^{-\varphi} \zeta \Phi^* u, \bar{\partial}^* v) &= (\bar{\partial}(e^{-\varphi} \zeta \Phi^* u), v) \\
&= (e^{-\varphi} (-\zeta \bar{\partial} \varphi \wedge \Phi^* u + \zeta \bar{\partial} \Phi^* \wedge u + \bar{\partial} \zeta \wedge \Phi^* u), v) \\
&= -(\bar{\partial} \varphi \wedge \Phi^* u, v)_{|\Phi|^{-2} \zeta} + (\bar{\partial} \Phi^* \wedge u, v)_{|\Phi|^{-2} \zeta} \\
&\quad + (e^{-\varphi} \Phi^* u, \text{grad}^{0,1} \zeta \lrcorner v) \\
&= -(\Phi^* u, \text{grad}^{0,1} \varphi \lrcorner v)_{|\Phi|^{-2} \zeta} + (\bar{\partial} \Phi^* \wedge u, v)_{|\Phi|^{-2} \zeta} \\
&\quad + (e^{-\varphi} \Phi^* u, \text{grad}^{0,1} \zeta \lrcorner v). \tag{24}
\end{aligned}$$

In the second equality, we used the condition $\bar{\partial} u = 0$. Now by substituting (15) in lemma 2.3 into (24), we have

$$(e^{-\varphi} \zeta \Phi^* u, \bar{\partial}^* v) = (e^{-\varphi} \Phi^* u, \text{grad}^{0,1} \zeta \lrcorner v) + (u, \text{Tr} B_\Phi A_v)_{|\Phi|^{-2} \zeta}. \tag{25}$$

Lemma 2.2 applied to the homomorphism

$$D = B_\Phi A_v \in \text{Hom}(E^*, \wedge^{n,k} M \otimes L \otimes E' \otimes E^*)$$

gives the following pointwise estimate.

$$\begin{aligned}
2|(u, \text{Tr} B_\Phi A_v)_{|\Phi|^{-2} \zeta}| &\leq \|u\|_{\frac{\zeta^2}{(\zeta+\delta)|\Phi|^2}}^2 + \|\sqrt{\zeta + \delta} \text{Tr} B_\Phi A_v\|_{|\Phi|^{-2}}^2 \\
&\leq \|u\|_{\frac{\zeta^2}{(\zeta+\delta)|\Phi|^2}}^2 + \left\| \sqrt{q(\zeta + \delta)} B_\Phi A_v \right\|_{|\Phi|^{-2}}^2. \tag{26}
\end{aligned}$$

Since

$$2\text{Re}(\text{grad}^{0,1} \zeta \lrcorner v, |\Phi|^{-2} \Phi^* u + \bar{\partial}^* v) \leq \left\| |\Phi|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_\tau^2 + \left\| \text{grad}^{0,1} \zeta \lrcorner v \right\|_{\tau^{-1}}^2,$$

from (21) (23) (25) (26), it follows that

$$\begin{aligned}
\left\| |\Phi|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_{\Omega, \zeta + \tau}^2 + \left\| \bar{\partial} v \right\|_{\Omega, \zeta}^2 &\geq \|u\|_{|\Phi|^{-2} \lambda \zeta}^2 - \|u\|_{\frac{\zeta^2}{(\zeta+\delta)|\Phi|^2}}^2 \\
&= \|u\|_{\frac{\zeta(\lambda\delta + \lambda\zeta - \zeta)}{(\zeta+\delta)|\Phi|^2}}^2. \tag{27}
\end{aligned}$$

This finishes the proof of the main estimate. □

4. A Division Theorem for Exact Sequences of Holomorphic Vector Bundles

In this section we give a sufficient integrability condition for the exactness at the level of global holomorphic sections for exact sequences of holomorphic vector bundles.

We consider a complex of holomorphic vector bundles over M ,

$$E \xrightarrow{\Phi} E' \xrightarrow{\Psi} E'' \quad (28)$$

i.e. $\Phi \in \Gamma(M, \text{Hom}(E, E')), \Psi \in \Gamma(M, \text{Hom}(E', E''))$ such that $\Psi \circ \Phi = 0$. E, E', E'' are assumed to be endowed with Hermitian structures.

We define for any $x \in M$

$$\mathcal{E}(x) = \min\{((\Psi^*\Psi + \Phi\Phi^*)\xi, \xi) \mid \xi \in E'_x, |\xi| = 1\} \quad (29)$$

where Φ^*, Ψ^* are the adjoint of Φ and Ψ respectively w.r.t. the given Hermitian structures. By definition, $0 \leq \mathcal{E} \in C(M)$ is the smallest eigenvalue of $\Psi^*\Psi + \Phi\Phi^*$. Suppose the complex (28) is exact at $x \in M$, let $\xi \in E'_x$ such that $(\Psi^*\Psi + \Phi\Phi^*)\xi = 0$, then by pairing with ξ we get $\Phi^*\xi = 0, \Psi\xi = 0$, i.e. $\xi \in \text{Ker}\Phi^* \cap \text{Ker}\Psi = \text{Im}\Phi^\perp \cap \text{Im}\Phi$ which implies $\xi = 0$. Conversely, we assume $\Psi^*\Psi + \Phi\Phi^*$ is an isomorphism on E'_x for some $x \in M$. Since $\text{Ker}\Psi$ is invariant under $\Psi^*\Psi + \Phi\Phi^*$, $\Psi^*\Psi + \Phi\Phi^*$ also induces an isomorphism on $\text{Ker}\Psi$. Let $\xi \in \text{Ker}\Psi$, there exists some $\eta \in \text{Ker}\Psi$ such that $\xi = (\Psi^*\Psi + \Phi\Phi^*)\eta = \Phi\Phi^*\eta \in \text{Im}\Phi$. Now we obtain the following useful fact about the function \mathcal{E} :

The complex (28) is exact at $x \in M$ if and only if $\mathcal{E}(x) > 0$.

When the complex (28) is exact, $\Phi^*(\Psi^*\Psi + \Phi\Phi^*)^{-1}|_{\text{Ker}\Psi}$ is a smooth lifting of Φ . So it is possible to establish division theorems by solving a coupled system consisting of

$$\bar{\partial}g = \bar{\partial}[\Phi^*(\Psi^*\Psi + \Phi\Phi^*)^{-1}f] \text{ and } \Phi g = 0$$

where $f \in \Gamma(E')$ satisfying $\Psi f = 0$. If g is a solution of this system, then $h \stackrel{\text{def}}{=} \Phi^*(\Psi^*\Psi + \Phi\Phi^*)^{-1}f - g \in \Gamma(E)$ and $\Phi h = f$. If Φ is surjective and E' is equipped with the quotient Hermitian structure then $\Psi = 0, \Phi\Phi^* = Id_{E'}$, and the above system reduces to

$$\bar{\partial}g = \bar{\partial}(\Phi^*f)$$

on the subbundle $\text{Ker}\Phi$. This key observation played an important role in both [S78] and [D82]. The difficulty of this method for our case is that $\text{Ker}\Phi$ is no longer a subbundle of E , so it amounts to solving $\bar{\partial}$ -equations for solutions valued in a subsheaf, it seems that it is not easy to give sufficient conditions for the solvability of this system.

The following lemma reduces our main theorem to the estimate (20). It was first formulated in [S72], the present version is quoted from [V08]).

Lemma4.1. Let H, H_0, H_1, H_2 be Hilbert spaces, $T : H_0 \rightarrow H$ be a bounded linear operator, $T_\ell : H_{\ell-1} \rightarrow H_\ell (\ell = 1, 2)$ be linear, closed, densely defined operators such that $T_2 \circ T_1 = 0$, and let $F \subseteq H$ be a closed subspace such that $T(\text{Ker}T_1) \subseteq F$. Then for every $f \in F$ the following statements are equivalent

1. there exists at least one $u \in \text{Ker}T_1$ and $C > 0$ such that $Tu = f$, $\|u\|_{H_0} \leq C$.
2. $|(g, f)_H|^2 \leq C^2(\|T^*g + T_1^*v\|_{H_0}^2 + \|T_2v\|_{H_2}^2)$ holds for any $g \in F, v \in \text{Dom}(T_1^*) \cap \text{Dom}(T_2)$.

The complex (28) is said to be generically exact if it is exact outside a subset of measure zero(w.r.t. dV_ω) of M .

Theorem4.2. Let (M, ω) be a Kähler manifold and let E, E', E'' be Hermitian holomorphic vector bundles over M , L a Hermitian line bundle over M . All the Hermitian structures may have singularities in a subvariety $Z \subsetneq M$ and $\Phi^{-1}(0) \subseteq Z$. Suppose that (28) is generically exact over M , $M \setminus Z$ is weakly pseudoconvex and that the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0, m \geq \min\{n - k + 1, r\}, 1 \leq k \leq n$;
2. the curvature of $\text{Hom}(E, E')$ satisfies

$$(F_{X\overline{X}}^{\text{Hom}(E, E')} \Phi, \Phi) \leq 0 \text{ for every } X \in T^{1,0}M;$$

3. the curvature of L satisfies

$$\sqrt{-1}(\varsigma c(L) - \partial\bar{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \bar{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma + \delta)\partial\bar{\partial}\varphi.$$

Then for every $\bar{\partial}$ -closed $(n, k-1)$ -form f which is valued in $L \otimes E'$ with $\Psi f = 0$ and $\|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}} < +\infty$, there exists a $\bar{\partial}$ -closed $(n, k-1)$ -form h valued in $L \otimes E$ such that $\Phi h = f$ and

$$\|h\|_{\frac{1}{\varsigma+\tau}} \leq \|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}}, \quad (30)$$

where $q = \max_{M \setminus Z} \text{rank} B_\Phi, \varphi = \log \|\Phi\|$, \mathcal{E} is the function defined by (29), $0 < \varsigma, \tau \in C^\infty(M)$ and δ is a measurable function on M satisfying $\mathcal{E}(\varsigma + \delta) \geq \|\Phi\|^2_\varsigma$.

Proof. Step 1. Let $\phi \in C^\infty(M \setminus Z)$ be a plurisubharmonic exhaustion function on $M \setminus Z$. For any $t > 0$, set $\Omega_t = \{x \in M \setminus Z | \phi(x) < t\}$. We know by definition $\Omega_t \Subset M \setminus Z$ and $\bigcup_t \Omega_t = M \setminus Z$.

Apart from a subset of \mathbb{R} in t which has measure zero, Ω_t is a pseudoconvex domain with smooth boundary, so our main estimate holds on such Ω_t . If we could find a $\bar{\partial}$ -closed section, say h_t , solving the equation $\Phi h_t = f$ on Ω_t with the estimate

$$\|h_t\|_{\Omega_t, \frac{1}{\varsigma+\tau}} \leq \|f\|_{\Omega_t, \frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}} \leq \|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}}.$$

By setting h_t to be zero outside Ω_t , we extend h_t to be an element of $L^2_{n,k-1}(M, L \otimes E, dV_{\frac{1}{\varsigma+\tau}})$. The above estimate allows us to apply the compactness argument to produce on M a $(n, k-1)$ -form h valued in $L \otimes E$ as the weak limit of $\{h_t\}$ in $L^2_{n,k-1}(M, L \otimes E, dV_{\frac{1}{\varsigma+\tau}})$ which satisfies the equation

$$\bar{\partial}h = 0 \text{ outside } Z. \quad (31)$$

and the following inequality

$$\|h\|_{\frac{1}{\varsigma+\tau}} \leq \lim_{t \rightarrow +\infty} \|h_t\|_{\Omega_t, \frac{1}{\varsigma+\tau}} \leq \|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}}.$$

By using the resulting L^2 -estimate and definition 3.2, we know h has L^2_{loc} (w.r.t the Lebesgue measure on the coordinate chart) coefficients under holomorphic frames over holomorphic coordinate charts. Since Z is an analytic subset of M , the extension lemma (D82, lemma 6.9) shows that $\bar{\partial}h = 0$ holds on the whole manifold M . Consequently, $\Phi h = f$ on M follows from solving $\Phi h_t = f$ on Ω_t with a L^2 -estimate for each $t > 0$.

Step 2. In order to make use of lemma 4.1, we introduce the following Hilbert spaces and densely defined operators.

$$\begin{aligned} H_0 &= L^2_{n,k-1}(\Omega, L \otimes E, dV_\omega), \quad H = L^2_{n,k-1}(\Omega, L \otimes E', dV_{|\Phi|^{-2}}), \\ H_1 &= L^2_{n,k}(\Omega, L \otimes E, dV_\omega), \quad H_2 = L^2_{n,k+1}(\Omega, L \otimes E, dV_\omega), \\ T &= \Phi \circ \sqrt{\varsigma + \tau} : H_0 \rightarrow H, \quad T_1 = \bar{\partial} \circ \sqrt{\varsigma + \tau} : H_0 \rightarrow H_1, \\ T_2 &= \sqrt{\varsigma} \circ \bar{\partial} : H_1 \rightarrow H_2, \end{aligned}$$

where $\Omega \Subset M \setminus Z$ is a domain satisfying conditions in lemma 3.3. Then $T : H_0 \rightarrow H$ is bounded (note that $\Omega \Subset M \setminus Z$), $T_\ell : H_{\ell-1} \rightarrow H_\ell$ ($\ell = 1, 2$) are closed, densely defined and satisfy $T_2 \circ T_1 = 0$. It is easy to see that the adjoint of T is given by

$$T^*u = \sqrt{\varsigma + \tau}|\Phi|^{-2}\Phi^*u, u \in H.$$

Similarly, from $0 < \varsigma, \tau \in C^\infty(M)$ we know

$$\text{Dom}T_1^* = \text{Dom}(\bar{\partial}^*), \text{Dom}T_2 = \text{Dom}(\bar{\partial})$$

and

$$T_1^*v = \sqrt{\varsigma + \tau}\bar{\partial}^*v, v \in \text{Dom}(T_1^*).$$

Define

$$F = \{u \in H \mid \Psi u = 0, \bar{\partial}u = 0\},$$

it is easy to see that $T(\text{Ker}T_1) \subseteq F$. Since Ψ and $\bar{\partial}$ are both closed operators, F is a closed subspace of H .

By the definition of \mathcal{E} , we know the following inequality

$$\begin{aligned} |\Phi^*u|^2 &= ((\Psi^*\Psi + \Phi\Phi^*)u, u) \\ &\geq \lambda|\Phi|^2|u|^2, \lambda = |\Phi|^{-2}\mathcal{E}. \end{aligned} \quad (32)$$

holds a.e.(w.r.t. dV_ω) on Ω for every $u \in F$.

Let $f \in A^{n,k-1}(L \otimes E')$ which is $\bar{\partial}$ -closed and satisfies $\Psi f = 0$ then we know by definition $f \in F$. Since $\mathcal{E}(\varsigma + \delta) \geq |\Phi|^2\varsigma$, we have

$$\lambda\delta + \lambda\varsigma \geq \varsigma.$$

From the a priori estimate (20) and the density lemma, we obtain the following inequality

$$\begin{aligned} |(u, f)_H|^2 &\leq \|f\|^2 \frac{\varsigma + \delta}{(\lambda\delta + \lambda\varsigma - \varsigma)\varsigma|\Phi|^2} \|u\|^2 \frac{\varsigma(\lambda\delta + \lambda\varsigma - \varsigma)}{(\varsigma + \delta)|\Phi|^2} \\ &\leq \|f\|^2 \frac{\varsigma + \delta}{(\lambda\delta + \lambda\varsigma - \varsigma)\varsigma|\Phi|^2} (\| |\Phi|^{-2}\Phi^*u + \bar{\partial}^*v \|_{\varsigma + \tau}^2 + \|\bar{\partial}v\|_\varsigma^2) \\ &= \|f\|^2 \frac{\varsigma + \delta}{(\lambda\delta + \lambda\varsigma - \varsigma)\varsigma|\Phi|^2} (\| \sqrt{\varsigma + \tau}|\Phi|^{-2}\Phi^*u + \sqrt{\varsigma + \tau}\bar{\partial}^*v \|^2 \\ &\quad + \| \sqrt{\varsigma}\bar{\partial}v \|^2) \\ &= \|f\|^2 \frac{\varsigma + \delta}{(\lambda\delta + \lambda\varsigma - \varsigma)\varsigma|\Phi|^2} (\|T^*u + T_1^*v\|_{H_0}^2 + \|T_2v\|_{H_2}^2) \end{aligned}$$

holds for any $u \in F, v \in \text{Dom}(T_1^*) \cap \text{Dom}(T_2)$. Note that the condition $\lambda\delta + \lambda\varsigma \geq \varsigma$ is needed for the first inequality. Hence we know by lemma 5.1 that there exist at least one $h' \in \text{Ker}T_1$ such that

$$Th' = f \text{ and } \|h'\|_{H_0}^2 \leq \|f\|^2 \frac{\varsigma + \delta}{(\lambda\delta + \lambda\varsigma - \varsigma)\varsigma|\Phi|^2}.$$

Letting $h = \sqrt{\varsigma + \tau}h'$, we have

$$\Phi h = f, \|h\|_{\frac{1}{\varsigma+\tau}}^2 \leq \|f\|^2_{\frac{\varsigma+\delta}{(\lambda\delta+\lambda\varsigma-\varsigma)\varsigma|\Phi|^2}}.$$

Replacing λ by $|\Phi|^{-2}\mathcal{E}$ completes the proof. \square

Remarks. (i) If M is weakly pseudoconvex and $Z = \eta^{-1}(0) \subsetneq M$ where η is a holomorphic function on M , then $M \setminus Z$ is weakly pseudoconvex. Let $\psi \in C^\infty(M)$ be a plurisubharmonic exhaustion function on M . It is easy to see that $\phi := \psi + |\eta|^{-1}$ is a plurisubharmonic exhaustion function on $M \setminus Z$. (ii) If M is a Stein manifold (or more generally, an essentially Stein manifold, see [V08]) and Z is an analytic hypersurface, then $M \setminus Z$ is a Stein manifold (or an essentially Stein manifold). (iii) When Φ is not identically zero, one can always find an analytic hypersurface Z such that $\Phi^{-1}(0) \subseteq Z$.

If we choose ς to be a positive constant, the third condition in theorem 4.2 will be independent of the function τ . By this observation, we have the following corollary.

Corollary 4.3. If the condition 3 in theorem 4.2 is replaced by

$$\sqrt{-1}c(L) \geq \sqrt{-1}q(|\Phi|^2\mathcal{E}^{-1} + 1)\partial\bar{\partial}\varphi, \quad (33)$$

then for every $\bar{\partial}$ -closed $(n, k-1)$ -form f which is valued in $L \otimes E'$ with

$$\Psi f = 0 \text{ and } \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}} < +\infty$$

there is a $\bar{\partial}$ -closed $(n, k-1)$ -form h valued in $L \otimes E$ such that $\Phi h = f$ and the following estimate holds

$$\|h\| \leq \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}}. \quad (34)$$

Proof. Set

$$\varsigma = 1, \tau = \text{constant} > 0, \delta = |\Phi|^2\mathcal{E}^{-1},$$

it is easy to see that $\mathcal{E}(\varsigma + \delta) \geq |\Phi|^2\varsigma$.

Hence, we get from theorem 4.2 that for every $\bar{\partial}$ -closed $(n, k-1)$ -form f which is valued in $L \otimes E'$ and satisfies

$$\Psi f = 0, \|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}} = \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}} < +\infty,$$

there is at least a $\bar{\partial}$ -closed $(n, k-1)$ -form h_τ valued in $L \otimes E$ such that

$$\Phi h_\tau = f \text{ and } \|h_\tau\|_{\frac{1}{1+\tau}} \leq \|f\|_{\frac{\varsigma+\delta}{(\varsigma+\delta)\varsigma\mathcal{E}-|\Phi|^2\varsigma^2}} = \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}}.$$

From the estimate given above, it follows that

$$\|h_\tau\| = \sqrt{1+\tau} \|h_\tau\|_{\frac{1}{1+\tau}} \leq \sqrt{1+\tau} \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}}.$$

The above estimate shows that $\{h_\tau\}_{1>\tau>0}$ bounded in $L^2_{n,k-1}(M, L \otimes E, dV_\omega)$, so we get a weak limit h of $\{h_\tau\}_{\tau>0}$ in $L^2_{n,k-1}(M, L \otimes E, dV_\omega)$ when $\tau \rightarrow 0$. It is easy to see that the resulting section h is $\bar{\partial}$ -closed on M and $\Phi h = f$. The L^2 -estimate of h_τ implies that

$$\begin{aligned} \|h\| &\leq \liminf_{\tau \rightarrow 0} \|h_\tau\| \\ &\leq \liminf_{\tau \rightarrow 0} \sqrt{1+\tau} \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}} \\ &= \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}} \end{aligned}$$

which completes the proof of corollary 4.3. □

Remarks. (i) The condition $(F_{X\overline{X}}^{\text{Hom}(E,E')} \Phi, \Phi) \leq 0$ is needed to handle the second term in (11). We recall that the curvature of the Chern connection of a Hermitian holomorphic vector bundle is semi-negative in the sense of Griffiths(Nakano) if and only if it is 1-tensor($\min\{n, r\}$ -tensor) semi-negative. Hence a sufficient condition for $(F_{X\overline{X}}^{\text{Hom}(E,E')} \Phi, \Phi) \leq 0$ is given by(since we always assume $E \geq_m 0$ for some positive integer m): E' is semi-negative in the sense of Griffiths.

(ii) In particular, if the underlying manifold M is assumed to be strongly pseudoconvex(corollary 4.5 (ii)) then one can always endow the holomorphic vector bundles over M , E and E' with Hermitian structures such that E is semi-positive in the sense of Nakano and E' is semi-negative in the sense of Griffiths. So our curvature conditions 1 and 2 are satisfied automatically by such Hermitian structures.

(iii) For these homomorphisms in the Koszul complex, due to the identity (50), the condition $(F_{X\overline{X}}^{\text{Hom}(E,E')} \Phi, \Phi) \leq 0$ holds provided the E is semi-positive in sense of Griffiths. The Koszul complex provides a series of homomorphisms which are not generically surjective. The generically surjective case has been extensively investigated in [S78] and [D82].

We can derive from corollary 4.3 the following results.

Corollary 4.4. Results in corollary 4.3 hold with the condition 2 assumed there replaced by the condition that E' is semi-negative in the sense of Griffiths.

Corollary 4.5. Besides the conditions in corollary 4.3, we also assume that (28) is exact on the whole manifold M . Then we have

(i) For every $\bar{\partial}$ -closed $(n, k-1)$ -form f which is valued in $L \otimes E'$ and locally square-integrable on M , if $\Psi f = 0$ then there exists a $\bar{\partial}$ -closed $h \in L^2_{n, k-1}(M, L \otimes E, dV_\omega)$ such that $\Phi h = f$. In particular, if E is semi-positive in the sense of Nakano, then the induced sequence on global section

$$\Gamma(M, K_M \otimes L \otimes E) \rightarrow \Gamma(M, K_M \otimes L \otimes E') \rightarrow \Gamma(M, K_M \otimes L \otimes E'') \quad (35)$$

is exact where K_M is the canonical bundle of M .

Moreover, if Φ is surjective then it induces a surjective homomorphism on cohomology groups:

$$\Phi : H^{n, k-1}(M, L \otimes E) \rightarrow H^{n, k-1}(M, L \otimes E'). \quad (36)$$

(ii) If (M, ω) is strongly pseudoconvex and $\Phi \in \Gamma(M, \text{Hom}(E, E'))$ is nonvanishing then (i) holds without assuming the curvature conditions 1-3.

Proof. The proof consisting of using appropriate weight functions to modify the given Hermitian structure on L to control the L^2 -norm and curvature.

(i) Let $0 < \phi \in \text{PSH}(M) \cap C^\infty(M)$ be an exhaustion function on M . Set $\Omega_t = \{z \in M | \phi(z) < t\}, t \in \mathbb{R}$ then $\Omega_t \Subset M$ and $\bigcup_t \Omega_t = M$. Since (28) is assumed to be exact on M , then we have $\mathcal{E}(x) > 0$ for every $x \in M$. Given $f \in A^{n, k-1}(L \otimes E')$, we can define a positive number for each $\ell = 0, 1, 2, \dots$

$$\delta_\ell = \sup \left\{ \frac{\mathcal{E}(x) + |\Phi(x)|^2}{\mathcal{E}(x)^2} |x \in \Omega_{\ell+1} \setminus \Omega_\ell \right\} \int_{\Omega_{\ell+1} \setminus \Omega_\ell} |f|^2 dV_\omega \in [0, +\infty).$$

We choose an increasing convex function $\eta \in C^\infty(\mathbb{R})$ such that

$$\eta(\ell) \geq \log(2^\ell \delta_\ell) \text{ for } \ell = 0, 1, 2, \dots \quad (37)$$

and set $\psi = \eta \circ \phi$. Then $\psi \in \text{PSH}(M) \cap C^\infty(M)$ and $e^{-\psi}h_L$ defines a singular Hermitian structure on the line bundle L where we denote by h_L the given Hermitian structure on L .

It is easy to see that on $M \setminus \Phi^{-1}(0)$ the curvature of $e^{-\psi}h_L$ satisfies

$$\sqrt{-1}c(L, e^{-\psi}h_L) = \sqrt{-1}(\partial\bar{\partial}\psi + c(L, h_L) \geq \sqrt{-1}q(|\Phi|^2\mathcal{E}^{-1} + 1)\partial\bar{\partial}\varphi.$$

By the construction of ψ , we get the following estimate of the L^2 -norm of f where the left hand side is computed by using the new Hermitian structure $e^{-\psi}h_L$ on L .

$$\begin{aligned} \|f\|_{\frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2}}^2 &= \int_M |f|^2 \frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2} e^{-\psi} dV_\omega \\ &= \sum_{\ell \geq 0} \int_{\Omega_{\ell+1} \setminus \Omega_\ell} |f|^2 \frac{\mathcal{E}+|\Phi|^2}{\mathcal{E}^2} e^{-\psi} dV_\omega \\ &\stackrel{(37)}{\leq} \sum_{\ell \geq 0} 2^{-\ell} = 2 < +\infty. \end{aligned} \tag{38}$$

From (38) and corollary 4.3, we get a $\bar{\partial}$ -closed section $h \in L_{n,k-1}^2(M, L \otimes E, dV_\omega)$ such that $\Phi h = f$ provided $\bar{\partial}f = 0$. Consequently, by using the De Rham-Weil isomorphism theorem, we know that if $\Psi = 0$ then the induced homomorphism $\Phi : H^{n,k-1}(M, L \otimes E) \rightarrow H^{n,k-1}(M, L \otimes E')$ is surjective. In the case of $k = 1$, our condition 1 becomes that E is nonnegative in the sense of Nakano. From the ellipticity of $\bar{\partial}$ (also due to the condition that $k = 1$), it follows that the induced sequence (35) is still exact.

(ii) As M is strongly pseudoconvex, one can modify the given Hermitian structures for E and E' such that E is semi-positive in the sense of Nakano and E' is semi-negative in the sense of Griffiths. With such Hermitian structures, we get the desired curvature conditions 1 and 2. Next, we multiply the given Hermitian structure on L by certain weight to make the new Hermitian structure satisfy condition 3. Let $\phi \in C^\infty(M)$ be a strictly plurisubharmonic exhaustion function of M .

Set

$\lambda(x) :=$ the smallest eigenvalue of $\sqrt{-1}\partial\bar{\partial}\phi(x)$ w.r.t. the metric ω ,

$\mu(x) :=$ the smallest eigenvalue of $\sqrt{-1}c(L, h_L)(x)$ w.r.t. the metric ω ,

$\Lambda(x) :=$ the largest eigenvalue of $\sqrt{-1}\partial\bar{\partial}\varphi(x)$ w.r.t. the metric ω ,

then it is easy to see that $\lambda, \mu, \Lambda \in C(M)$ and

$$\sqrt{-1}\partial\bar{\partial}\phi \geq \lambda\omega, \sqrt{-1}c(L, h_L) \geq \mu\omega, \sqrt{-1}\partial\bar{\partial}\varphi \leq \Lambda\omega.$$

Since ϕ is strictly plurisubharmonic, we know $\lambda > 0$ on M . We can therefore define a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$\sigma(t) = \sup\left\{\frac{q(|\Phi|^2(x)\mathcal{E}(x)^{-1} + 1)\Lambda(x) - \mu(x)}{\lambda(x)} \mid x \in \Omega_t\right\}, t \in \mathbb{R}. \quad (39)$$

For this $\sigma(t)$, one can always find a $\chi \in C^\infty[0, +\infty)$ such that

$$\chi'(t) \geq \max\{\sigma(t), 0\}, \chi''(t) \geq 0, t > 0. \quad (40)$$

Now endow the line bundle L with a new Hermitian structure $e^{-\psi_1}h_L$ where $\psi_1 = \chi \circ \phi$. It is obvious that

$$\begin{aligned} \sqrt{-1}c(L, e^{-\psi_1}h_L) &= \sqrt{-1}(\partial\bar{\partial}\psi_1 + c(L, h_L)) \\ &= \sqrt{-1}(\chi' \circ \phi \partial\bar{\partial}\phi + \chi'' \circ \phi \partial\phi \wedge \bar{\partial}\phi + c(L, h_L)) \\ &\stackrel{(40)}{\geq} \chi' \circ \phi \lambda\omega + \sqrt{-1}c(L, h_L) \\ &\stackrel{\lambda > 0}{\geq} \sigma \circ \phi \lambda\omega + \sqrt{-1}c(L, h_L) \\ &\stackrel{(39)}{\geq} (q(|\Phi|^2\mathcal{E}^{-1} + 1)\Lambda - \mu)\omega + \sqrt{-1}c(L, h_L) \\ &\geq \sqrt{-1}q(|\Phi|^2\mathcal{E}^{-1} + 1)\partial\bar{\partial}\varphi. \end{aligned} \quad (41)$$

Conclusion (ii) follows from (41) and conclusion (i), this finishes the proof of corollary 4.5. \square

Given a holomorphic section $\Phi \in \Gamma(M, \text{Hom}(E, E'))$, there is the following exact sequence of sheaves over M

$$\mathcal{O}(E) \rightarrow \mathcal{O}(E') \rightarrow \mathcal{O}(E')/\text{Im}\Phi \rightarrow 0$$

where $\text{Im}\Phi$ is the image of the induced sheaf-homomorphism $\Phi : \mathcal{O}(E) \rightarrow \mathcal{O}(E')$. Generally, the quotient sheaf $\mathcal{O}(E')/\text{Im}\Phi$ is never locally free, so this case does not fit into the general setting established by theorem 4.2, corollary 4.3 and corollary 4.4.

However we can modify definition of the function \mathcal{E} to make the same argument works for this situation. To this end, we have to introduce the following function \mathcal{E}_1 (instead of \mathcal{E}) by using a single homomorphism Φ . We define for any $x \in M$,

$$\mathcal{E}_1(x) = \min\{((\Psi^*\Psi + \Phi\Phi^*)\xi, \xi) \mid \xi \in E'_x, |\xi| = 1\} \quad (29)'$$

where Ψ is the orthogonal projection from E'_x onto the subspace $\Phi(E_x)^\perp$. It is obvious that \mathcal{E}_1 is positive everywhere. From the fact that $|\Psi\xi| = \inf\{|\xi + \Phi(\eta)| \mid \eta \in E_x\}$ for every $\xi \in E'_x$ and our definition (29)' we know the function \mathcal{E}_1 is upper semi-continuous and therefore measurable.

Similar to theorem 4.2, we have the following result about the division problem for a single holomorphic homomorphism.

Theorem 4.2'. Let (M, ω) be a Kähler manifold and let E, E', E'' be Hermitian holomorphic vector bundles over M , L a Hermitian line bundle over M . All the Hermitian structures may have singularities in a subvariety $Z \subsetneq M$ and $\Phi^{-1}(0) \subseteq Z$. Suppose that $M \setminus Z$ is weakly pseudoconvex and that the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0, m \geq \min\{n - k + 1, r\}, 1 \leq k \leq n$;
2. the curvature of $\text{Hom}(E, E')$ satisfies

$$(F_{X\overline{X}}^{\text{Hom}(E, E')} \Phi, \Phi) \leq 0 \text{ for every } X \in T^{1,0}M;$$

3. the curvature of L satisfies

$$\sqrt{-1}(\varsigma c(L) - \partial\bar{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \bar{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma + \delta)\partial\bar{\partial}\varphi.$$

Then for every $\bar{\partial}$ -closed $(n, k - 1)$ -form f which is valued in $L \otimes E'$ and satisfies

$$f(x) \in \Phi(E_x) \text{ for a.e. } x \in M \text{ and } \|f\|_{\frac{\varsigma + \delta}{(\varsigma + \delta)\varsigma\mathcal{E}_1 - |\Phi|^2\varsigma^2}} < +\infty,$$

there exists a $\bar{\partial}$ -closed $(n, k - 1)$ -form h valued in $L \otimes E$ such that $\Phi h = f$ and $\|h\|_{\frac{1}{\varsigma + \tau}} \leq \|f\|_{\frac{\varsigma + \delta}{(\varsigma + \delta)\varsigma\mathcal{E}_1 - |\Phi|^2\varsigma^2}}$, where $q = \max_{M \setminus Z} \text{rank} B_\Phi, \varphi = \log \|\Phi\|^2, \mathcal{E}_1$ is the function defined by (29)', $0 < \varsigma, \tau \in C^\infty(M)$ and δ is a measurable function on M satisfying $\mathcal{E}_1(\varsigma + \delta) \geq \|\Phi\|^2\varsigma$.

Remark. The results parallel to corollaries 4.3-4.4 can be easily derived from theorem 4.2'. Since the function \mathcal{E}_1 defined by (29)' is only upper semi-continuous, it can't be locally bounded from below by positive constants. So we don't have the result parallel to corollary 4.5.

5. Applications to Koszul Complex

In this section we apply results obtained in section 4 to the special case of generalized Koszul complex.

Let M be a complex manifold and E be a holomorphic vector bundle of rank r over M . The Koszul complex associated to a section $s \in \Gamma(E^*)$ is defined as follows

$$\det E \xrightarrow{d_r} \wedge^{r-1} E \xrightarrow{d_{r-1}} \dots \xrightarrow{d_1} \mathcal{O}_M \xrightarrow{d_0} 0 \quad (42)$$

where the boundary operators are given by the interior product

$$d_p = s \lrcorner, 1 \leq p \leq r. \quad (43)$$

It forms a complex since we have $d_{p-1} \circ d_p = 0$ for $1 \leq p \leq r$.

In particular, if we set $E = T^{*1,0}M$ and $s = X$, a holomorphic vector field on M , then the complex (42) is given by

$$K_M \xrightarrow{X \lrcorner} \wedge^{n-1} T^{*1,0}M \xrightarrow{X \lrcorner} \dots \rightarrow T^{*1,0}M \xrightarrow{X \lrcorner} \mathcal{O}_M \xrightarrow{X \lrcorner} 0$$

which recovers the usual notion of the Koszul complex associated to a vector field X on a complex manifold M .

As before, in order to handle the curvature term in the Bochner formula we consider the following complex associated to (42):

$$L \otimes \det E \xrightarrow{d_r} L \otimes \wedge^{r-1} E \xrightarrow{d_{r-1}} \dots \xrightarrow{d_1} L \xrightarrow{d_0} 0,$$

where L is a holomorphic line bundle over M .

We start with improving the estimate in lemma 3.3 for

$$\Phi = s \lrcorner \in \Gamma(M, \text{Hom}(\wedge^p E, \wedge^{p-1} E))$$

$1 \leq p \leq r$. In the following discussion, E and L are endowed with Hermitian structures. Let $\{e_1, \dots, e_r\}$ be a local holomorphic frame of E and let $\{e_1^*, \dots, e_r^*\}$ be its dual frame. We conclude from the definition

$$\Phi^* = \theta \wedge$$

where

$$\theta = \bar{g}_i h^{\bar{i}j} e_j \text{ and } s = g_i e_i^*. \quad (44)$$

By choosing a local frame $\{e_1, \dots, e_r\}$ normal at a given point $x \in M \setminus s^{-1}(0)$ such that

$$e_1^*(x) = \frac{s}{|s|}(x),$$

then we get

$$\begin{aligned} \|\Phi\|^2(x) &= \sum_{i_1 < \dots < i_p} |\Phi(e_{i_1} \wedge \dots \wedge e_{i_p})|^2 \\ &= \sum_{1 < i_2 < \dots < i_p} \|s|e_{i_2} \wedge \dots \wedge e_{i_p}|^2 = \binom{r}{p-1} |s|^2(x). \end{aligned} \quad (45)$$

Since for any $\xi \in \wedge^{p-1} E_x, x \in M$, we have

$$\theta \wedge s \lrcorner \xi + s \lrcorner \theta \wedge \xi = |s|^2 \xi,$$

so the function \mathcal{E} (in (29)) is given by

$$\mathcal{E}(x) = |s(x)|^2.$$

This implies that the complex (42) is exact at $x \in M$ if and only if $s(x) \neq 0$.

We will denote by B_s the second fundamental form of the line bundle in E^* generated by s over $M \setminus s^{-1}(0)$, i.e.

$$B_s(X) = (\nabla_X^{E^*} s)^\perp \quad (46)$$

where ∇^{E^*} is the Chern connection on E^* , $X \in T_x^{1,0} M, x \in M \setminus s^{-1}(0)$.

Lemma 5.1. We have the following relations between s and the associated homomorphism $\Phi = s \lrcorner$:

$$B_\Phi = B_{s \lrcorner} \quad (47)$$

$$\|B_\Phi A\|^2 = \binom{r}{p-1} \|B_s A\|^2 \quad (48)$$

$$\text{Tr} B_\Phi A = \text{Tr} B_s A \quad (49)$$

$$(F_{X\overline{X}}^{\text{Hom}(\wedge^p E, \wedge^{p-1} E)} \Phi, \Phi) = \binom{r}{p-1} (F_{X\overline{X}}^{E^*} s, s) \quad (50)$$

where $X \in T_x^{1,0}M$, $x \in M \setminus s^{-1}(0)$, $A \in \text{Hom}(\wedge^{n,k-1}TM \otimes L^* \otimes E^*, T^{1,0}M)$, $\text{Tr}B_s A$ is defined by (12) with ρ being the interior product and F^{E^*} is the curvature of the induced Chern connection on E^* , $1 \leq k \leq n$.

Proof. We first choose holomorphic coordinates and frames $\{z_1, \dots, z_n\}$, $\{e_1, \dots, e_r\}$, $\{\sigma\}$ which are normal at a given point $x \in M \setminus s^{-1}(0)$.

For every $\xi \in \wedge^p E_x$ we have by the definition of B

$$\begin{aligned} B(X) \cdot \xi &= \nabla_X^{\text{Hom}(\wedge^p E, \wedge^{p-1} E)}(\Phi) \cdot \xi \\ &= \nabla_X^{\wedge^{p-1} E}(\Phi \xi) - \Phi(\nabla_X^{\wedge^p E} \xi) \\ &= \nabla_X^{\wedge^{p-1} E}(s \lrcorner \xi) - s \lrcorner (\nabla_X^{\wedge^p E} \xi) = (\nabla_X^{E^*} s) \lrcorner \xi. \end{aligned}$$

Combining this equality with (45) and using the assumption that $\{e_1, \dots, e_r\}$ is normal at x , we obtain

$$\begin{aligned} (B(X), \Phi) &= (B(X) \cdot e_{i_1} \wedge \dots \wedge e_{i_p}, \Phi(e_{i_1} \wedge \dots \wedge e_{i_p})) \\ &= (\nabla_X^{E^*} s \lrcorner (e_{i_1} \wedge \dots \wedge e_{i_p}), s \lrcorner (e_{i_1} \wedge \dots \wedge e_{i_p})) \\ &= (\nabla_X^{\wedge^{p-1} E}(s \lrcorner (e_{i_1} \wedge \dots \wedge e_{i_p})), s \lrcorner (e_{i_1} \wedge \dots \wedge e_{i_p})) \\ &= X(|s \lrcorner (e_{i_1} \wedge \dots \wedge e_{i_p})|^2) \\ &= X(|\Phi|^2) = \binom{r}{p-1} X(|s|^2). \end{aligned}$$

Now it follows from definition (8) that

$$\begin{aligned} B_\Phi(X) \cdot \xi &= B(X) \cdot \xi - (B(X), \Phi) \frac{\Phi(\xi)}{|\Phi|^2} \\ &= (\nabla_X^{E^*} s - \frac{X(|s|^2)}{|s|^2} s) \lrcorner \xi. \\ &= (\nabla_X^{E^*} s - \frac{(\nabla_X^* s, s)}{|s|^2} s) \lrcorner \xi = B_s(X) \lrcorner \xi. \end{aligned}$$

For increasing multi-indices K, I with $|K| = k-1, |I| = p$, we denote $X_{\overline{KI}} = A(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_{i_1}^* \wedge \dots \wedge e_{i_p}^*) \in T_x^{1,0}M$, then we get from (47) that

$$\begin{aligned} \|B_\Phi A\|^2 &= \left\| B_\Phi A \left(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_{i_1}^* \wedge \dots \wedge e_{i_p}^* \right) \right\|^2 \\ &= \|B_\Phi(X_{\overline{KI}})\|^2 \\ &= \|B_\Phi(X_{\overline{KI}}) \cdot (e_{j_1} \wedge \dots \wedge e_{j_p})\|^2 \\ &\stackrel{(47)}{=} \|B_s(X_{\overline{KI}}) \lrcorner (e_{j_1} \wedge \dots \wedge e_{j_p})\|^2 \\ &= \binom{r}{p-1} \|B_s(X_{\overline{KI}})\|^2 = \binom{r}{p-1} \|B_s A\|^2. \end{aligned}$$

By (47) and the definition (12), we have

$$\begin{aligned} \text{Tr}B_\Phi A &= (B_\Phi A \left(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_{i_1}^* \wedge \dots \wedge e_{i_p}^* \right), \\ &\quad e_{j_1} \wedge \dots \wedge e_{j_{p-1}} \otimes e_{i_1}^* \wedge \dots \wedge e_{i_p}^*) dz \otimes \sigma \otimes e_{j_1} \wedge \dots \wedge e_{j_{p-1}} \\ &= (B_\Phi(X_{i_1 \dots i_p}) \cdot (e_{i_1} \wedge \dots \wedge e_{i_p}), e_{j_1} \wedge \dots \wedge e_{j_{p-1}}) \\ &\quad \cdot dz \otimes \sigma \otimes e_{j_1} \wedge \dots \wedge e_{j_{p-1}} \\ &\stackrel{(47)}{=} (B_s(X_{i_1 \dots i_p}) \lrcorner (e_{i_1} \wedge \dots \wedge e_{i_p}), e_{j_1} \wedge \dots \wedge e_{j_{p-1}}) \\ &\quad \cdot dz \otimes \sigma \otimes e_{j_1} \wedge \dots \wedge e_{j_{p-1}} \\ &= (B_s(X_{i_1 \dots i_p}) \lrcorner (dz \otimes \sigma \otimes e_{i_1} \wedge \dots \wedge e_{i_p})) \end{aligned}$$

$$\begin{aligned}
&= (B_s(A(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*))) \\
&\quad \lrcorner (dz \otimes \sigma \otimes e_{i_1} \wedge \cdots \wedge e_{i_p}) \\
&= \text{Tr} B_s A.
\end{aligned}$$

We denote by F, F^p the curvatures of E and $\wedge^p E$ respectively ($1 \leq p \leq r$). Let $\xi = \frac{1}{p!} \sum_{1 \leq i_1, \dots, i_p \leq r} \xi_{i_1 \dots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$, $X = X_\alpha \frac{\partial}{\partial z_\alpha}$. In all of the computations involving ξ during the proof of the equality (50), sum is always taken over all repeated indices (not only over strictly increasing multi-indices). We know by definition that

$$\begin{aligned}
(F_{X\bar{X}}^p \xi, \xi) &= \frac{1}{p!} (F_{X\bar{X}}^p \xi)_{i_1 \dots i_p} \overline{\xi_{i_1 \dots i_p}} \\
&= \frac{1}{p!} \sum_{1 \leq a \leq p} F_{\alpha \bar{\beta} i_a \bar{j}} X_\alpha \xi_{i_1 \dots (i)_{a \dots i_p}} \overline{X_\beta \xi_{i_1 \dots i_p}} \\
&= \frac{1}{p!} \sum_{1 \leq a \leq p} F_{\alpha \bar{\beta} i_a \bar{j}} X_\alpha \xi_{ii_1 \dots \widehat{i_a} \dots i_p} \overline{X_\beta \xi_{i_a i_1 \dots \widehat{i_a} \dots i_p}} \\
&= \frac{1}{p!} \sum_{1 \leq a \leq p} F_{\alpha \bar{\beta} i_j \bar{j}} X_\alpha \xi_{ji_1 \dots \widehat{i_a} \dots i_p} \overline{X_\beta \xi_{ii_1 \dots \widehat{i_a} \dots i_p}}. \tag{51}
\end{aligned}$$

where $\widehat{i_a}$ means omitting the index i_a . Similarly, we have

$$\begin{aligned}
(F_{X\bar{X}}^{p-1} \Phi \xi, \Phi \xi) &= \frac{1}{(p-1)!} F_{\alpha \bar{\beta} i_j \bar{j}} X_\alpha (\Phi \xi)_{ji_1 \dots \widehat{i_a} \dots i_{p-1}} \overline{X_\beta (\Phi \xi)_{ii_1 \dots \widehat{i_a} \dots i_{p-1}}} \\
&= \sum_{1 \leq a \leq p-1} F_{\alpha \bar{\beta} i_j \bar{j}} X_\alpha g_k \xi_{jki_1 \dots \widehat{i_a} \dots i_{p-1}} \overline{X_\beta g_l \xi_{ili_1 \dots \widehat{i_a} \dots i_{p-1}}}.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
(\Phi F_{X\bar{X}}^p \xi, \Phi \xi) &= \frac{1}{(p-1)!} (\Phi F_{X\bar{X}}^p \xi)_{i_1 \dots i_{p-1}} \overline{(\Phi \xi)_{i_1 \dots i_{p-1}}} \\
&= \frac{1}{(p-1)!} g_k (F_{X\bar{X}}^p \xi)_{ki_1 \dots i_{p-1}} \overline{g_l \xi_{li_1 \dots i_{p-1}}} \\
&= \frac{1}{(p-1)!} F_{\alpha \bar{\beta} k \bar{i}} X_\alpha g_k \xi_{ii_1 \dots i_{p-1}} \overline{g_l \xi_{li_1 \dots i_{p-1}}} \\
&\quad + \frac{1}{(p-1)!} \sum_{1 \leq a \leq p-1} F_{\alpha \bar{\beta} i_a \bar{j}} X_\alpha g_k \xi_{ki_1 \dots (j)_{a \dots i_{p-1}}} \overline{X_\beta g_l \xi_{li_1 \dots i_{p-1}}} \\
&= \frac{1}{(p-1)!} F_{\alpha \bar{\beta} k \bar{i}} X_\alpha g_k \xi_{ii_1 \dots i_{p-1}} \overline{X_\beta g_l \xi_{li_1 \dots i_{p-1}}} \\
&\quad + \frac{1}{(p-1)!} \sum_{1 \leq a \leq p-1} F_{\alpha \bar{\beta} i_a \bar{j}} X_\alpha g_k \xi_{jki_1 \dots \widehat{i_a} \dots i_{p-1}} \overline{X_\beta g_l \xi_{i_a li_1 \dots \widehat{i_a} \dots i_{p-1}}} \\
&= \frac{1}{(p-1)!} F_{\alpha \bar{\beta} k \bar{i}} X_\alpha g_k \xi_{ii_1 \dots i_{p-1}} \overline{X_\beta g_l \xi_{li_1 \dots i_{p-1}}} \\
&\quad + \frac{1}{(p-1)!} \sum_{1 \leq a \leq p-1} F_{\alpha \bar{\beta} i_j \bar{j}} X_\alpha g_k \xi_{jki_1 \dots \widehat{i_a} \dots i_{p-1}} \overline{X_\beta g_l \xi_{ii_1 \dots \widehat{i_a} \dots i_{p-1}}}.
\end{aligned}$$

Form the last two equalities, it follows that

$$\begin{aligned}
((F_{X\bar{X}}^{\text{Hom}(\wedge^p E, \wedge^{p-1} E)} \Phi) \xi, \Phi \xi) &= (F_{X\bar{X}}^{p-1} \Phi \xi, \Phi \xi) - (\Phi F_{X\bar{X}}^p \xi, \Phi \xi) \\
&= -\frac{1}{(p-1)!} F_{\alpha \bar{\beta} k \bar{i}} X_\alpha g_k \xi_{ii_1 \dots i_{p-1}} \\
&\quad \cdot \overline{X_\beta g_l \xi_{li_1 \dots i_{p-1}}}
\end{aligned}$$

which implies

$$(F_{X\bar{X}}^{\text{Hom}(\wedge^p E, \wedge^{p-1} E)} \Phi, \Phi) = \sum_{i_1 < \dots < i_p} (F_{X\bar{X}}^{\text{Hom}(\wedge^p E, \wedge^{p-1} E)} \Phi \cdot e_{i_1} \wedge \cdots \wedge e_{i_p}$$

$$\begin{aligned}
& , \Phi \cdot e_{i_1} \wedge \cdots \wedge e_{i_p}) \\
& = - \sum_{i_1 < \cdots < i_{p-1}} F_{\alpha \bar{\beta} k \bar{l}} X_{\alpha} g_k \overline{X_{\beta} g_l} \operatorname{sgn} \begin{pmatrix} i & i_1 & \cdots & i_{p-1} \\ l & i_1 & \cdots & i_{p-1} \end{pmatrix} \\
& = - \binom{r}{p-1} F_{\alpha \bar{\beta} k \bar{l}} X_{\alpha} g_k \cdot \overline{X_{\beta} g_l} = \binom{r}{p-1} (F_{X \bar{X}}^{E*} s, s).
\end{aligned}$$

The proof is complete. \square

Consequently, we obtain the following identities.

Lemma5.2. For any $u \in \wedge^{n,k-1} M \otimes L \otimes \wedge^{p-1} E$ and $v \in \wedge^{n,k} M \otimes L \otimes \wedge^p E$, $1 \leq k \leq n$, we have the following pointwise identities outside $s^{-1}(0)$.

$$\partial_{\alpha} \partial_{\bar{\beta}} \psi v_{\alpha \bar{K} I} \overline{v_{\beta \bar{K} I}} = e^{-\psi} \|B_s A_v\|^2 - e^{-\psi} (F_{X_{\bar{K} i}}^{E*} \overline{X_{\bar{K} i}} s, s). \quad (52)$$

$$(\bar{\partial} \Phi^* \wedge u, v) - (\Phi^* u, \operatorname{grad}^{0,1} \psi \lrcorner v) = (u, \operatorname{Tr} B_s A_v) \quad (53)$$

where $\psi = \log |s|^2$, A_v is defined by (9), $\Phi = s \lrcorner$, $v = v_{\bar{K} I} dz \otimes d\bar{z}_K \otimes \sigma \otimes e_I$, $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$, and $X_{\bar{K} i} = A_v(\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}_K} \otimes \sigma^* \otimes e_i^*)$.

Proof. Let $\varphi = \log |\Phi|^2$, then we know, by (45), $\varphi = \psi + \log \binom{r}{p-1}$. From the identity (11), it follows that

$$\begin{aligned}
\partial_{\alpha} \partial_{\bar{\beta}} \psi v_{\alpha \bar{K} I} \overline{v_{\beta \bar{K} I}} &= \partial_{\alpha} \partial_{\bar{\beta}} \varphi v_{\alpha \bar{K} I} \overline{v_{\beta \bar{K} I}} \\
&= e^{-\varphi} [\|B_{\Phi} A_v\|^2 - (F_{X_{\bar{K} i} \overline{X_{\bar{K} i}}}^{\operatorname{Hom}(E, E')} \Phi, \Phi)] \\
&\stackrel{(48), (50)}{=} e^{-\psi} \|B_s A_v\|^2 - e^{-\psi} (F_{X_{\bar{K} i} \overline{X_{\bar{K} i}}}^{E*} s, s).
\end{aligned}$$

(53) follows from (15) (45) (47). The proof is finished. \square

Now we improve, for $\Phi = s \lrcorner$, the main estimate obtained in section 3.

Lemma5.3. Let (M, ω) be a Kähler manifold and let E be a Hermitian holomorphic vector bundle over M , L a Hermitian holomorphic line bundle over M . $\Omega \Subset M \setminus s^{-1}(0)$ is a pseudoconvex domain with smooth boundary. Suppose that the following conditions hold on Ω .

1. $E \geq_m 0$, $m \geq \min\{n - k + 1, r - p + 1\}$, $1 \leq k \leq n$, $1 \leq p \leq r$;
2. the curvature of L satisfies

$$\sqrt{-1}(\zeta c(L) - \partial \bar{\partial} \zeta - \tau^{-1} \partial \zeta \wedge \bar{\partial} \zeta) \geq \sqrt{-1} q(\zeta + \delta) \partial \bar{\partial} \psi.$$

Then the following estimate

$$\left\| |s|^{-2} \theta \wedge u + \bar{\partial}^* v \right\|_{\Omega, \varsigma + \tau}^2 + \left\| \bar{\partial} v \right\|_{\Omega, \varsigma}^2 \geq \|u\|_{\Omega, \frac{\varsigma \delta}{(\varsigma + \delta)|s|^2}}^2 \quad (54)$$

holds for every $\bar{\partial}$ -closed $u \in A^{n, k-1}(\bar{\Omega}, L \otimes \wedge^{p-1} E)$ satisfying $s \lrcorner u = 0$ and every $v \in A^{n, k}(\bar{\Omega}, L \otimes \wedge^p E) \cap \text{Dom}(\bar{\partial}^*)$, where θ is defined in (44), $\psi = \log |s|^2$, $q = \min\{n, r-1\}$, $n = \dim_{\mathbb{C}} M$, $r = \text{rank}_{\mathbb{C}} E$, $0 < \varsigma \in C^\infty(\Omega)$ and δ, τ are measurable functions on Ω satisfying $\tau > 0, \varsigma + \delta \geq 0$.

Proof. The proof is essentially the same as that of lemma 3.3, we sketch it with an emphasis on the modifications.

By the following identity

$$\begin{aligned} |\Phi^* u|^2 &= (\theta \wedge u, \theta \wedge u) = (s \lrcorner \theta \wedge u, u) \\ &= (s \lrcorner \theta \wedge u + \theta \wedge s \lrcorner u, u) = (|s|^2 u, u) = |s|^2 |u|^2, \end{aligned}$$

we obtain

$$\begin{aligned} \text{l.h.s. of (54)} &= \left\| |s|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_{\varsigma + \tau}^2 + \left\| \bar{\partial} v \right\|_{\varsigma}^2 \\ &= \|u\|_{|s|^{-2}\varsigma}^2 + \left\| |s|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_{\tau}^2 \\ &\quad + 2\text{Re}(\varsigma e^{-\psi} \Phi^* u, \bar{\partial}^* v) + \left\| \sqrt{\varsigma} \bar{\partial}^* v \right\|^2 + \left\| \sqrt{\varsigma} \bar{\partial} v \right\|^2. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} ([\sqrt{-1} F^{L \otimes \wedge^p E}, \Lambda_\omega] v, v) &= F_{\alpha\beta I J}^{L \otimes \wedge^p E} v_{\alpha K, I} \overline{v_{\beta K, J}} \\ &= F_{\alpha\beta I J}^{\wedge^p E} v_{\alpha K, I} \overline{v_{\beta K, J}} + F_{\alpha\beta}^L v_{\alpha K, I} \overline{v_{\beta K, I}} \\ &= F_{\alpha\beta i j}^E v_{\alpha K, i N} \overline{v_{\beta K, i N}} + F_{\alpha\beta}^L v_{\alpha K, I} \overline{v_{\beta K, I}} \\ &\geq F_{\alpha\beta}^L v_{\alpha K, I} \overline{v_{\beta K, I}}, \end{aligned}$$

the last inequality follows from the condition $E \geq_m 0, m \geq \min\{n-k+1, r-p+1\}$. Now we get by using the twisted Bochner-Kodaira-Nakano formula and Morrey's trick (to handle the boundary term)

$$\begin{aligned} \text{l.h.s. of (54)} &\geq \|u\|_{|s|^{-2}\varsigma}^2 + \left\| |s|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_{\tau}^2 \\ &\quad + \int_{\Omega} \varsigma F_{\alpha\beta}^L v_{\alpha K, I} \overline{v_{\beta K, I}} - \nabla^{\bar{\alpha}} \nabla^{\beta} \varsigma \left(\frac{\partial}{\partial \bar{z}_{\alpha}} \lrcorner v, \frac{\partial}{\partial \bar{z}_{\beta}} \lrcorner v \right) \\ &\quad + 2\text{Re}(\varsigma e^{-\psi} \Phi^* u + \text{grad}^{0,1} \varsigma \lrcorner v, \bar{\partial}^* v) dV_{\omega} \\ &\stackrel{(51)}{\geq} \|u\|_{|s|^{-2}\varsigma}^2 + \left\| |s|^{-2} \Phi^* u + \bar{\partial}^* v \right\|_{\tau}^2 \\ &\quad + \int_{\Omega} \varsigma F_{\alpha\beta}^L v_{\alpha K, I} \overline{v_{\beta K, I}} - \nabla^{\bar{\alpha}} \nabla^{\beta} \varsigma \left(\frac{\partial}{\partial \bar{z}_{\alpha}} \lrcorner v, \frac{\partial}{\partial \bar{z}_{\beta}} \lrcorner v \right) \\ &\quad + 2\text{Re}(\varsigma e^{-\psi} \Phi^* u + \text{grad}^{0,1} \varsigma \lrcorner v, \bar{\partial}^* v) dV_{\omega} \end{aligned}$$

$$\begin{aligned}
&\geq \|u\|_{|s|^{-2}\zeta}^2 + \left\| |s|^{-2}\Phi^*u + \bar{\partial}^*v \right\|_{\tau}^2 \\
&\quad + \int_{\Omega} (q(\zeta + \delta)\partial_{\alpha}\partial_{\bar{\beta}}\psi + \tau^{-1}\partial_{\alpha}\zeta\partial_{\bar{\beta}}\zeta)v_{\alpha\bar{K},I}\overline{v_{\beta\bar{K},I}} \\
&\quad + 2\operatorname{Re}(\zeta e^{-\psi}\Phi^*u + \operatorname{grad}^{0,1}_{\zeta}\lrcorner v, \bar{\partial}^*v)dV_{\omega} \\
&\stackrel{(52)}{\geq} \|u\|_{|s|^{-2}\zeta}^2 + \left\| |s|^{-2}\Phi^*u + \bar{\partial}^*v \right\|_{\tau}^2 + \left\| \sqrt{q(\zeta + \delta)}B_sA_v \right\|_{|s|^{-2}}^2 \\
&\quad + \left\| \operatorname{grad}^{0,1}_{\zeta}\lrcorner v \right\|_{\tau^{-1}}^2 + 2\operatorname{Re}(\zeta e^{-\psi}\Phi^*u + \operatorname{grad}^{0,1}_{\zeta}\lrcorner v, \bar{\partial}^*v) \quad (55)
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
2\operatorname{Re}(\zeta e^{-\psi}\Phi^*u, \bar{\partial}^*v) &= 2\operatorname{Re}(e^{-\psi}\Phi^*u, \operatorname{grad}^{0,1}_{\zeta}\lrcorner v) + 2\operatorname{Re}(\bar{\partial}\Phi^* \wedge u, v)_{|s|^{-2}\zeta} \\
&\quad - 2\operatorname{Re}(\Phi^*u, \operatorname{grad}^{0,1}_{\psi}\lrcorner v)_{|s|^{-2}\zeta} \\
&\stackrel{(53)}{=} 2\operatorname{Re}(e^{-\psi}\Phi^*u, \operatorname{grad}^{0,1}_{\zeta}\lrcorner v) + 2\operatorname{Re}(u, \operatorname{Tr}B_sA_v)_{|s|^{-2}\zeta} \\
&\geq 2\operatorname{Re}(e^{-\psi}\Phi^*u, \operatorname{grad}^{0,1}_{\zeta}\lrcorner v) \\
&\quad - \left\| \sqrt{\zeta + \delta}\operatorname{Tr}B_sA_v \right\|_{|s|^{-2}}^2 - \|u\|_{\frac{\zeta^2}{(\zeta + \delta)|s|^2}}^2. \quad (56)
\end{aligned}$$

By definition, we have

$$\operatorname{rank}B_sA_v \leq \operatorname{rank}B_s \leq \min\{n, r - 1\}.$$

It follows from (13) (55) and (56) that

$$\begin{aligned}
\text{l.h.s. of (55)} &\geq \|u\|_{\frac{\zeta\delta}{(\zeta + \delta)|s|^2}}^2 + \left\| |s|^{-2}\Phi^*u + \bar{\partial}^*v \right\|_{\tau}^2 \\
&\quad + \left\| \operatorname{grad}^{0,1}_{\zeta}\lrcorner v \right\|_{\tau^{-1}}^2 + 2\operatorname{Re}(\operatorname{grad}^{0,1}_{\zeta}\lrcorner v, e^{-\psi}\Phi^*u + \bar{\partial}^*v) \\
&\geq \|u\|_{\frac{\zeta\delta}{(\zeta + \delta)|s|^2}}^2,
\end{aligned}$$

which completes the proof. \square

By the standard functional argument used in the proof of theorem 4.2 with the estimate (20) replaced by the improved one (54), we obtain the main result of this section:

Theorem 5.4. Let (M, ω) be a Kähler manifold and let E be a Hermitian holomorphic vector bundle over M , L a line bundle over M , $s \in \Gamma(E^*)$. All the Hermitian structures may have singularities in a subvariety $Z \subsetneq M$. We define the Koszul complex associated to s by (42) (43). Assume that $s^{-1}(0) \subseteq Z$, and that $M \setminus Z$ is weakly pseudoconvex and that the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0$, $m \geq \min\{n - k + 1, r - p + 1\}$, $1 \leq k \leq n, 1 \leq p \leq n$;
2. the curvature of L satisfies

$$\sqrt{-1}(\varsigma c(L) - \partial\bar{\partial}\varsigma - \tau^{-1}\partial\varsigma \wedge \bar{\partial}\varsigma) \geq \sqrt{-1}q(\varsigma + \delta)\partial\bar{\partial}\varphi.$$

Then for any $\bar{\partial}$ -closed $(n, k-1)$ -form f which is valued in $L \otimes \wedge^{p-1}E$, if $d_{p-1}f = 0$ and $\|f\|_{\frac{\varsigma+\delta}{\varsigma\delta|s|^2}} < +\infty$ then there is at least one $\bar{\partial}$ -closed $(n, k-1)$ -form h valued in $L \otimes \wedge^pE$ such that $d_ph = f$ and the following estimate holds

$$\|h\|_{\frac{1}{\varsigma+\tau}} \leq \|f\|_{\frac{\varsigma+\delta}{\varsigma\delta|s|^2}}, \quad (57)$$

where $1 \leq p \leq r$, $\varphi = \log|s|$, $q = \min\{n, r-1\}$, $n = \dim_{\mathbb{C}} M$, $r = \text{rank}_{\mathbb{C}} E$, $0 < \varsigma, \tau \in C^\infty(M)$ and $\delta \geq 0$ is a measurable function on M .

We can derive from theorem 5.4 the next result by repeating the argument used in the proof of corollary 4.3.

Corollary 5.5. Let (M, ω) be a Kähler manifold and let E be a Hermitian holomorphic vector bundle over M , L a line bundle over M , $s \in \Gamma(E^*)$. All the Hermitian structures may have singularities in a subvariety $Z \subsetneq M$. We define the Koszul complex associated to s by (42) (43). Assume that $s^{-1}(0) \subseteq Z$, and that $M \setminus Z$ is weakly pseudoconvex and the following conditions hold on $M \setminus Z$:

1. $E \geq_m 0$, $m \geq \min\{n-k+1, r-p+1\}$, $1 \leq k \leq n$, $1 \leq p \leq r$;
2. the curvature of L satisfies $\sqrt{-1}c(L) \geq \sqrt{-1}q(1+\varepsilon)\partial\bar{\partial}\varphi$.

Then for any $\bar{\partial}$ -closed $(n, k-1)$ -form f valued in $L \otimes \wedge^{p-1}E$, if $d_{p-1}f = 0$ and $\|f\|_{|s|^{-2}} < +\infty$ then there is at least one $\bar{\partial}$ -closed $(n, k-1)$ -form h valued in $L \otimes \wedge^pE$ such that $d_ph = f$ with the estimate

$$\|h\|^2 \leq \frac{1+\varepsilon}{\varepsilon} \|f\|_{|s|^{-2}}^2, \quad (58)$$

where $1 \leq p \leq r$, $\varphi = \log|s|^2$, $q = \min\{n, r-1\}$, $n = \dim_{\mathbb{C}} M$, $r = \text{rank}_{\mathbb{C}} E$ and ε is a positive constant.

As a consequence of corollary 4.5, we also have the following sufficient condition for the exactness of the induced sequence of global sections.

Corollary 5.6. Let (M, ω) be a weakly pseudoconvex Kähler manifold and let E be a Hermitian holomorphic vector bundle over M , L a line bundle over M . Assume that $s \in \Gamma(E^*)$ is a nonvanishing section and that

1. E is semi-positive in the sense of Nakano;
2. the curvature of L satisfies

$$\sqrt{-1}c(L) \geq \sqrt{-1}q(1 + \varepsilon)\partial\bar{\partial}\varphi \text{ for some positive constant } \varepsilon.$$

Then the induced sequence on global sections

$$\Gamma(K_M \otimes L \otimes \det E) \xrightarrow{d_r} \Gamma(K_M \otimes L \otimes \wedge^{r-1} E) \xrightarrow{d_{r-1}} \dots \xrightarrow{d_1} \Gamma(K_M \otimes L) \xrightarrow{d_0} 0$$

is exact.

Now we discuss the special case of Koszul complex over a domain $\Omega \subseteq \mathbb{C}^n$.

Let $g_1, \dots, g_r \in \mathcal{O}(\Omega)$, the Koszul complex associated to $g = (g_1, \dots, g_r)$ is given by

$$\wedge^r \mathcal{O}^{\oplus r} \xrightarrow{d_r} \wedge^{r-1} \mathcal{O}^{\oplus r} \xrightarrow{d_{r-1}} \dots \xrightarrow{d_2} \wedge \mathcal{O}^{\oplus r} \xrightarrow{d_1} \mathcal{O} \xrightarrow{d_0} 0 \quad (59)$$

where the boundary operators are defined by $d_p = g_{\lrcorner}, 1 \leq p \leq r$.

It is easy to see that for every $h = (h_{i_1 \dots i_p})_{i_1 \dots i_p=1}^r \in \Gamma(\Omega, \wedge^p \mathcal{O}^{\oplus r})$ (i.e. $h_{i_1 \dots i_p} \in \mathcal{O}(\Omega)$ and $h_{i_1 \dots i_p}$ is skew symmetric in i_1, \dots, i_p), we have

$$d_p h = (f_{i_1 \dots i_{p-1}})_{i_1 \dots i_{p-1}=1}^r \in \Gamma(\Omega, \wedge^{p-1} \mathcal{O}^{\oplus r}) \text{ with } f_{i_1 \dots i_{p-1}} = \sum_{1 \leq \nu \leq r} g_{\nu} h_{\nu i_1 \dots i_{p-1}}.$$

Given a measurable function ϕ on Ω which is locally bounded from above, we can define the following space

$$\left\{ (h_{i_1 \dots i_p})_{i_1 \dots i_p=1}^r \in \Gamma(\Omega, \wedge^p \mathcal{O}^{\oplus r}) \mid \sum_{i_1 < \dots < i_p} \int_{\Omega} |h|^2 e^{-\phi} dV < +\infty \right\}$$

where $|h|^2 := \sum_{i_1 < \dots < i_p} |h_{i_1 \dots i_p}|^2$. This space will be denoted by $A_{\Omega}^2(\wedge^p \mathcal{O}^{\oplus r}, \phi)$,

$1 \leq p \leq r$. Since the measurable function ϕ is locally bounded from above, we know that $A_{\Omega}^2(\wedge^p \mathcal{O}^{\oplus r}, \phi)$ is a Hilbert space.

As a consequence of corollary 5.5, it follows that

Corollary 5.7. Suppose Ω is a pseudoconvex domain in \mathbb{C}^n , $\psi \in \text{PSH}(\Omega)$, $g_1, \dots, g_r \in \mathcal{O}(\Omega)$, $1 \leq p \leq r$, and $\varepsilon > 0$ is a constant. For every cycle f of the Koszul complex of holomorphic functions, if $f \in A_{\Omega}^2(\wedge^{p-1} \mathcal{O}^{\oplus r}, \psi + (q + q\varepsilon + 1) \log |g|)$, then there exists at least one holomorphic $h \in A_{\Omega}^2(\wedge^p \mathcal{O}^{\oplus r}, \psi + q(1 + \varepsilon) \log |g|^2)$ such that f is the image of h under the boundary map and

$$\int_{\Omega} |h|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi} dV \leq \frac{1+\varepsilon}{\varepsilon} \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV. \quad (60)$$

where $|g| = \sqrt{\sum_i |g_i|^2}$, $q = \min\{n, r-1\}$.

Particularly, if $|g| \neq 0$ holds on Ω then (59) induces an exact sequence on global sections:

$$\Gamma(\Omega, \wedge^r \mathcal{O}^{\oplus r}) \xrightarrow{d_r} \Gamma(\Omega, \wedge^{r-1} \mathcal{O}^{\oplus r}) \xrightarrow{d_{r-1}} \dots \xrightarrow{d_2} \Gamma(\Omega, \wedge \mathcal{O}^{\oplus r}) \xrightarrow{d_1} \Gamma(\Omega, \mathcal{O}) \xrightarrow{d_0} 0.$$

Proof. Step 1. We first give a proof in the case of $\psi \in \text{PSH}(\Omega) \cap C^\infty(\Omega)$. Let $E = \mathcal{O}^{\oplus r}$, $L = \mathcal{O}$, $s = (g_1, \dots, g_r) \in \Gamma(E^*)$. We define the Hermitian structure on E to be h_0 , the standard Hermitian structure on $\mathcal{O}^{\oplus r}$ induced from \mathbb{C}^r , so (E, h_0) is flat.

Let $\varepsilon > 0$, then the following function

$$\frac{1}{(\sum_i |g_i|^2)^{q(1+\varepsilon)} e^\psi}$$

defines a Hermitian structure on L which has singularity in $Z := g_1^{-1}(0)$ (without loss of generality, we assume g_1 is not identically zero). The curvature this Hermitian structure is given by

$$\begin{aligned} \sqrt{-1}c(L) &= \sqrt{-1}\partial\bar{\partial}(\log(\sum_i |g_i|^2)^{-q(1+\varepsilon)} e^{-\psi}) \\ &= q(1+\varepsilon)\sqrt{-1}\partial\bar{\partial}\log|s|^2 + \sqrt{-1}\partial\bar{\partial}\psi \\ &\geq q(1+\varepsilon)\sqrt{-1}\partial\bar{\partial}\log|s|^2. \end{aligned}$$

So far we have checked that all of the conditions assumed in corollary 5.5 are fulfilled by the Hermitian structures on E, L and $s \in \Gamma(E^*)$ as constructed above. Hence the desired solvability and estimate in this case follow from corollary 5.5.

Step 2. Now we proceed to prove the general case where ψ is only assumed to be plurisubharmonic on Ω .

Let $\Omega_1 \Subset \Omega_2 \Subset \dots$ be a pseudoconvex exhaustion of Ω , and let $\psi_\ell \in \text{PSH}(\Omega_\ell) \cap C^\infty(\Omega_\ell)$ for $\ell \geq 1$ such that

$$\psi_\ell \searrow \psi \text{ as } \ell \nearrow \infty \text{ on every } \Omega_j.$$

Fix some $j \geq 1$, without loss of generality, we could assume $\psi_\ell \in \text{PSH}(\Omega_j) \cap C^\infty(\Omega_j)$, $\ell \geq 1$. Since $\psi_\ell \geq \psi$ holds on Ω_j , we get

$$\int_{\Omega_j} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi_\ell} dV \leq \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV < +\infty$$

which implies (by the result proved in step 1) that there exists a holomorphic $h_{j\ell} \in A_{\Omega_j}^2(\wedge^p \mathcal{O}^{\oplus r}, \psi_\ell + q(1+\varepsilon)\log|g|^2)$ such that

$$d_p h_{j\ell} = f|_{\Omega_j}$$

and

$$\begin{aligned} \int_{\Omega_j} |h_{j\ell}|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi_1} dV &\leq \int_{\Omega_j} |h_{j\ell}|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi_\ell} dV \\ &\leq \frac{1+\varepsilon}{\varepsilon} \int_{\Omega_j} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi_\ell} dV \\ &\leq \frac{1+\varepsilon}{\varepsilon} \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV. \end{aligned}$$

By the above estimate, we know $\{h_{j\ell}\}$ is bounded in $A_{\Omega_j}^2(\wedge^p \mathcal{O}^{\oplus r}, \psi_1 + q(1+\varepsilon) \log |g|)$. Consequently we can find a weak limit h_j of $\{h_{j\ell}\}$ as $\ell \rightarrow \infty$ in $A_{\Omega_j}^2(\wedge^p \mathcal{O}^{\oplus r}, \psi_1 + q(1+\varepsilon) \log |g|)$.

It is easy to see that

$$d_p h_j = f|_{\Omega_j}$$

and

$$\begin{aligned} \int_{\Omega_j} |h_j|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi_1} dV &\leq \lim_{\ell \rightarrow \infty} \int_{\Omega_j} |h_{j\ell}|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi_1} dV \\ &\leq \frac{1+\varepsilon}{\varepsilon} \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV. \end{aligned}$$

From the resulting estimate on h_j , we can repeat the same argument of taking a weak limit of $\{h_j\}$ when $j \rightarrow \infty$ and then use the standard Cantor diagonalization process to show the existence of the desired section h , it finishes our proof of corollary 5.7. \square

The special case of $p = 1$ in corollary 5.7 is Skoda's division theorem.

Corollary 5.8. Suppose Ω is a pseudoconvex domain in \mathbb{C}^n , $\psi \in \text{PSH}(\Omega)$, $g_1 \cdots, g_r \in \mathcal{O}(\Omega)$, and $\varepsilon > 0$ is a constant, then for every $f \in A_{\Omega}^2(\psi + (q + q\varepsilon + 1) \log |g|^2)$, there exist holomorphic functions $h_1, \cdots, h_r \in A_{\Omega}^2(\psi + q(1 + \varepsilon) \log |g|^2)$ such that $f = \sum_i g_i h_i$ and

$$\int_{\Omega} |h|^2 |g|^{-2q(1+\varepsilon)} e^{-\psi} dV \leq \frac{1+\varepsilon}{\varepsilon} \int_{\Omega} |f|^2 |g|^{-2(q+q\varepsilon+1)} e^{-\psi} dV. \quad (61)$$

where $|g|^2 = \sum_i |g_i|^2$, $|h|^2 = \sum_i |h_i|^2$, $q = \min\{n, r-1\}$.

We end up this section by giving a sufficient condition which is deduced from Skoda's division theorem by purely algebraic argument.

Let Ω be a domain in \mathbb{C}^n , and Φ be a $q \times p$ matrix of holomorphic functions on Ω , $p \geq q$. We denote by $\delta_{i_1 \cdots i_q}$ the $q \times q$ minors of Φ , i.e.

$$\delta_{i_1 \dots i_q} = \det \begin{pmatrix} \Phi_{1i_1} & \dots & \Phi_{1i_q} \\ \vdots & \ddots & \vdots \\ \Phi_{qi_1} & \dots & \Phi_{qi_q} \end{pmatrix},$$

where $1 \leq i_1 < i_2 < \dots < i_q \leq p$. There are $\binom{p}{q}$ distinct minors of order q .

Proposition 5.9. Let $\psi \in \text{PSH}(\Omega)$, $f \in \mathcal{O}^q(\Omega)$, if Ω is pseudoconvex and there exists a constant $\alpha > 1$ such that

$$\int_{\Omega} \frac{|f|^2}{\left(\sum_{i_1 < \dots < i_q} |\delta_{i_1 \dots i_q}|^2 \right)^{\beta}} e^{-\psi} dV < +\infty, \quad (62)$$

where $\beta = \min\{n, \binom{p}{q} - 1\} \cdot \alpha + 1$. Then there is at least one $h \in \mathcal{O}^p(\Omega)$ which solves the equations $\Phi h = f$.

Proof. Let f_1, \dots, f_q be the components of f , since for each $1 \leq \nu \leq q$ we have

$$\int_{\Omega} \frac{|f_{\nu}|^2}{\left(\sum_{i_1 < \dots < i_q} |\delta_{i_1 \dots i_q}|^2 \right)^{\beta}} e^{-\psi} dV \leq \int_{\Omega} \frac{|f|^2}{\left(\sum_{i_1 < \dots < i_q} |\delta_{i_1 \dots i_q}|^2 \right)^{\beta}} e^{-\psi} dV < +\infty,$$

there exists, by Skoda's theorem (i.e. corollary 5.8), a system of functions $u_{i_1 \dots i_q, \nu} \in \mathcal{O}^q(\Omega)$, $1 \leq i_1 < i_2 < \dots < i_q \leq p$, such that

$$f_{\nu} = \sum_{i_1 < \dots < i_q} \delta_{i_1 \dots i_q} u_{i_1 \dots i_q, \nu}.$$

Set

$$u_{i_1 \dots i_q} = \begin{pmatrix} u_{i_1 \dots i_q, 1} \\ \vdots \\ u_{i_1 \dots i_q, q} \end{pmatrix},$$

then the above equality could be rewritten as

$$f = \sum_{i_1 < \dots < i_q} \delta_{i_1 \dots i_q} u_{i_1 \dots i_q}. \quad (63)$$

For fixed $1 \leq i_1 < i_2 < \dots < i_q \leq p$, we consider the p by p matrix $\Phi_{i_1 \dots i_q}$ whose entries are defined by

$$\Phi_{i_1 \dots i_q, ij} = \begin{cases} \Phi_{ij}, & \text{if } 1 \leq i \leq q; \\ 1, & \text{if } q+1 \leq i \leq p, j = j_{i-q}; \\ 0, & \text{otherwise,} \end{cases} \quad (64)$$

where $1 \leq j_1 < j_2 < \dots < j_{p-q} \leq p$ are the indices complementary to $1 \leq i_1 < i_2 < \dots < i_q \leq p$.

It follows from the above definition of $\Phi_{i_1 \dots i_q}$ that

$$\det \Phi_{i_1 \dots i_q} = \text{sgn} \begin{pmatrix} 1 \dots q & q+1 \dots p \\ i_1 \dots i_q & j_1 \dots j_{p-q} \end{pmatrix} \delta_{i_1 \dots i_q}. \quad (65)$$

We define $\tilde{u}_{i_1 \dots i_q} \in \mathcal{O}^p(\Omega)$, $1 \leq i_1 < i_2 < \dots < i_q \leq p$, by

$$\tilde{u}_{i_1 \dots i_q} = \begin{pmatrix} u_{i_1 \dots i_q} \\ v_{i_1 \dots i_q} \end{pmatrix}, \quad (66)$$

where $v_{i_1 \dots i_q} \in \mathcal{O}^{p-q}(\Omega)$ could be an arbitrary section.

By using $\tilde{u}_{i_1 \dots i_q}$ we introduce an element of $\mathcal{O}^p(\Omega)$ as follows

$$h_{i_1 \dots i_q} = \Phi_{i_1 \dots i_q}^* \tilde{u}_{i_1 \dots i_q} \quad (67)$$

for $1 \leq i_1 < i_2 < \dots < i_q \leq p$. In the definition (67), $\Phi_{i_1 \dots i_q}^*$ is the adjoint matrix of $\Phi_{i_1 \dots i_q}$.

Multiplying (67) by the matrix $\Phi_{i_1 \dots i_q}$, then the identity (65) gives

$$\begin{aligned} \Phi_{i_1 \dots i_q} h_{i_1 \dots i_q} &= \det \Phi_{i_1 \dots i_q} \tilde{u}_{i_1 \dots i_q} \\ &= \text{sgn} \begin{pmatrix} 1 \dots q & q+1 \dots p \\ i_1 \dots i_q & j_1 \dots j_{p-q} \end{pmatrix} \delta_{i_1 \dots i_q} \tilde{u}_{i_1 \dots i_q} \end{aligned}$$

By definition (64) we know

$$\Phi_{i_1 \dots i_q} h_{i_1 \dots i_q} = \begin{pmatrix} \Phi h_{i_1 \dots i_q} \\ * \end{pmatrix},$$

so comparing the first q rows in the above equality and (66) shows that

$$\Phi h_{i_1 \dots i_q} = \text{sgn} \begin{pmatrix} 1 \dots q & q+1 \dots p \\ i_1 \dots i_q & j_1 \dots j_{p-q} \end{pmatrix} \delta_{i_1 \dots i_q} u_{i_1 \dots i_q}. \quad (68)$$

Now by using $h_{i_1 \dots i_q} \in \mathcal{O}^p(\Omega)$ given in (67), we set

$$h = \sum_{i_1 < \dots < i_q} \text{sgn} \begin{pmatrix} 1 \dots q & q+1 \dots p \\ i_1 \dots i_q & j_1 \dots j_{p-q} \end{pmatrix} h_{i_1 \dots i_q},$$

then from (63) and (68) we obtain

$$\begin{aligned}\Phi h &= \operatorname{sgn} \binom{1 \cdots q \quad q+1 \cdots p}{i_1 \cdots i_q \quad j_1 \cdots j_{p-q}} \Phi h_{i_1 \cdots i_q} \\ &\stackrel{(68)}{=} \sum_{i_1 < \cdots < i_q} \delta_{i_1 \cdots i_q} u_{i_1 \cdots i_q} \stackrel{(63)}{=} f.\end{aligned}$$

which completes the proof. □

Remark. In [KT71], a similar condition was used to characterize the membership for the elements of finitely generated submodules of $A_p^{\oplus m}$.

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